4.2.4. Given the integrable functions

$$f_1(x) = \sum_{j=0}^{\infty} 2^{-j} F_{2^{2^{2^j}}}(x)\,, \qquad f_2(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} F_{2^{2^j}}(x)\,, \qquad x \in \mathbf{T}^1,$$

show that $\|f_1 * D_N\|_{L^1} \to \infty$ and $\|f_2 * D_N\|_{L^1} \to \infty$ as $N \to \infty$.
$\big[$*Hint:* Let $M_j = 2^{2^{2^j}}$ or $M_j = 2^{2^j}$ depending on the situation. For fixed N let j_N be the least integer j such that $M_j > N$. Then for $j \geq j_N + 1$ we have $M_j \geq M_{j_N}^2 > N^2 \geq 2N+1$, hence $\frac{M_j - N}{M_j + 1} \geq \frac{1}{2}$. Split the summation indices into the sets $j \geq j_N$ and $j < j_N$. Conclude that $\|f_1 * D_N\|_{L^1}$ and $\|f_2 * D_N\|_{L^1}$ tend to infinity as $N \to \infty$ using Exercise 4.2.3.$\big]$

4.3 Multipliers, Transference, and Almost Everywhere Convergence

In Chapter 2 we saw that bounded operators from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ that commute with translations are given by convolution with tempered distributions on $\mathbf{R}^n$. In particular, when $p = q$, these tempered distributions have bounded Fourier transforms, called Fourier multipliers. Convolution operators that commute with translations can also be defined on the torus. These lead to Fourier multipliers on the torus.

4.3.1 Multipliers on the Torus

In analogy with the nonperiodic case, we could identify convolution operators on $\mathbf{T}^n$ with appropriate distributions on the torus; see Exercise 4.3.2 for an introduction to this topic. However, it is simpler to avoid this point of view and consider the study of multipliers directly, bypassing the discussion of distributions on the torus.

For $h \in \mathbf{T}^n$ we define the *translation operator* τ^h acting on a periodic function f as follows: $\tau^h(f)(x) = f(x-h)$ for $x \in \mathbf{T}^n$. We say that a linear operator T acting on functions on the torus *commutes with translations* if for all $h \in \mathbf{T}^n$ we have $\tau^h(T(f))(x) = T(\tau^h f)(x)$ for almost all $x \in \mathbf{T}^n$.

Theorem 4.3.1. *Suppose that T is a bounded linear operator that commutes with translations and maps $L^p(\mathbf{T}^n)$ to $L^q(\mathbf{T}^n)$ for some $1 \leq p, q \leq \infty$. Then there exists a bounded sequence $\{a_m\}_{m\in\mathbf{Z}^n}$ such that*

$$T(f)(x) = \sum_{m\in\mathbf{Z}^n} a_m \widehat{f}(m) e^{2\pi i m\cdot x} \tag{4.3.1}$$