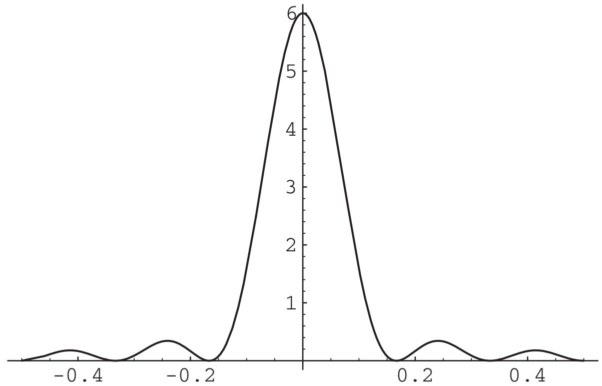

Fig. 1.2 The Fejér kernel F_5 plotted on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Example 1.2.18. On the circle group $\mathbf{T}^1$ let

$$F_N(t) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) e^{2\pi i j t} = \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)}\right)^2 . \tag{1.2.27}$$

To check the previous equality we use that

$$\sin^2(x) = (2 - e^{2ix} - e^{-2ix})/4 ,$$

and we carry out the calculation. F_N is called the *Fejér kernel*. See Figure 1.2. To see that the sequence $\{F_N\}_N$ is an approximate identity, we check conditions (i), (ii), and (iii) in Definition 1.2.15. Property (iii) follows from the expression giving F_N in terms of sines, while property (i) follows from the expression giving F_N in terms of exponentials. Property (ii) is identical to property (i), since F_N is nonnegative.

Next comes the basic theorem concerning approximate identities.

Theorem 1.2.19. *Let k_ε be an approximate identity on a locally compact group G with left Haar measure λ.*

(1) If f lies in $L^p(G)$ for $1 \le p < \infty$, then $\|k_\varepsilon * f - f\|_{L^p(G)} \to 0$ as $\varepsilon \to 0$.
(2) Let f be a function in $L^\infty(G)$ that is uniformly continuous on a subset K of G, in the sense that for all $\delta > 0$ there is a neighborhood V of the identity element such that for all $x \in K$ and $y \in V$ we have $|f(y^{-1}x) - f(x)| < \delta$. Then we have that $\|k_\varepsilon * f - f\|_{L^\infty(K)} \to 0$ as $\varepsilon \to 0$. In particular, if f is bounded and continuous at a point $x_0 \in G$, then $(k_\varepsilon * f)(x_0) \to f(x_0)$ as $\varepsilon \to 0$.

Proof. We start with the case $1 \le p < \infty$. We recall that continuous functions with compact support are dense in L^p of locally compact Hausdorff spaces equipped with measures arising from nonnegative linear functionals; see [152, Theorem 12.10]. For a continuous function g supported in a compact set L ~~we have~~ we have $|g(h^{-1}x) - g(x)|^p \le (2\|g\|_{L^\infty})^p \chi_{W^{-1}L}$ for h in a relatively compact neighborhood

W of the identity element e. By the Lebesgue dominated convergence theorem we obtain

$$\int_G |g(h^{-1}x) - g(x)|^p \, d\lambda(x) \to 0 \tag{1.2.28}$$

as $h \to e$. Now approximate a given f in $L^p(G)$ by a continuous function with compact support g to deduce that

$$\int_G |f(h^{-1}x) - f(x)|^p \, d\lambda(x) \to 0 \qquad \text{as} \qquad h \to e\,. \tag{1.2.29}$$

Because of (1.2.29), given a $\delta > 0$ there exists a neighborhood V of e such that

$$h \in V \implies \int_G |f(h^{-1}x) - f(x)|^p \, d\lambda(x) < \left(\frac{\delta}{2c}\right)^p, \tag{1.2.30}$$

where c is the constant that appears in Definition 1.2.15 (i). Since k_ε has integral one for all $\varepsilon > 0$, we have

$$\begin{aligned} (k_\varepsilon * f)(x) - f(x) &= (k_\varepsilon * f)(x) - f(x) \int_G k_\varepsilon(y) \, d\lambda(y) \\ &= \int_G (f(y^{-1}x) - f(x)) k_\varepsilon(y) \, d\lambda(y) \\ &= \int_V (f(y^{-1}x) - f(x)) k_\varepsilon(y) \, d\lambda(y) \\ &\qquad + \int_{V^c} (f(y^{-1}x) - f(x)) k_\varepsilon(y) \, d\lambda(y)\,. \end{aligned} \tag{1.2.31}$$

Now take L^p norms in x in (1.2.31). In view of (1.2.30),

$$\begin{aligned} &\left\| \int_V (f(y^{-1}x) - f(x)) k_\varepsilon(y) \, d\lambda(y) \right\|_{L^p(G, d\lambda(x))} \\ &\qquad \le \int_V \big\| f(y^{-1}x) - f(x) \big\|_{L^p(G, d\lambda(x))} |k_\varepsilon(y)| \, d\lambda(y) \\ &\qquad < \int_V \frac{\delta}{2c} |k_\varepsilon(y)| \, d\lambda(y) \le \frac{\delta}{2}\,, \end{aligned} \tag{1.2.32}$$

while

$$\begin{aligned} &\left\| \int_{V^c} (f(y^{-1}x) - f(x)) k_\varepsilon(y) \, d\lambda(y) \right\|_{L^p(G, d\lambda(x))} \\ &\qquad \le \int_{V^c} 2 \big\| f \big\|_{L^p(G)} |k_\varepsilon(y)| \, d\lambda(y) < \frac{\delta}{2}\,, \end{aligned} \tag{1.2.33}$$

provided we have that

$$\int_{V^c} |k_\varepsilon(x)| \, d\lambda(x) < \frac{\delta}{4\big(\|f\|_{L^p} + 1\big)}\,. \tag{1.2.34}$$

Choose $\varepsilon_0 > 0$ such that (1.2.34) is valid for $\varepsilon < \varepsilon_0$ by property (iii). Now (1.2.32) and (1.2.33) imply the required conclusion.

The case $p = \infty$ follows similarly. Let f be a bounded function on G that is uniformly continuous on K. Given $\delta > 0$, there is a neighborhood V of e such that, whenever $y \in V$ and $x \in K$ we have

$$|f(y^{-1}x) - f(x)| < \frac{\delta}{2c}, \tag{1.2.35}$$

where c is as in Definition 1.2.15 (i). By property (iii) in Definition 1.2.15, there is an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ we have

$$\int_{V^c} |k_\varepsilon(y)|\, d\lambda(y) < \frac{\delta}{4\big(\|f\|_{L^\infty(G)} + 1\big)}. \tag{1.2.36}$$

Using (1.2.35) and (1.2.36), we deduce that

$$\begin{aligned}
&\sup_{x\in K} |(k_\varepsilon * f)(x) - f(x)| \\
&\le \int_V |k_\varepsilon(y)| \sup_{x\in K} |f(y^{-1}x) - f(x)|\, d\lambda(y) + \int_{V^c} |k_\varepsilon(y)| \sup_{x\in K} |f(y^{-1}x) - f(x)|\, d\lambda(y) \\
&\le c\,\frac{\delta}{2c} + \frac{\delta}{4\big(\|f\|_{L^\infty(G)} + 1\big)}\, 2\|f\|_{L^\infty(G)} \le \delta.
\end{aligned}$$

This shows that $k_\varepsilon * f$ converge uniformly to f on K as $\varepsilon \to 0$. In particular, if $K = \{x_0\}$ and f is bounded and continuous at x_0, we have $(k_\varepsilon * f)(x_0) \to f(x_0)$. $\square$

Remark 1.2.20. Observe that if Haar measure satisfies (1.2.12), then the conclusion of Theorem 1.2.19 also holds for $f * k_\varepsilon$.

A simple modification in the proof of Theorem 1.2.19 yields the following variant, which presents a significant difference only when $a = 0$.

Theorem 1.2.21. *Let k_ε be a family of functions on a locally compact group G that satisfies properties (i) and (iii) of Definition 1.2.15 and also*

$$\int_G k_\varepsilon(x)\, d\lambda(x) = a$$

for some fixed $a \in \mathbf{C}$ and for all $\varepsilon > 0$. Let $f \in L^p(G)$ for some $1 \le p \le \infty$.

(a) If $1 \le p < \infty$, then $\|k_\varepsilon * f - af\|_{L^p(G)} \to 0$ as $\varepsilon \to 0$.
(b) If $p = \infty$ and f is uniformly continuous on a subset K of G, in the sense that for any $\delta > 0$ there is a neighborhood V of the identity element of G such that $\sup_{x\in K} \sup_{y\in V} |f(y^{-1}x) - f(x)| \le \delta$, then we have that $\|k_\varepsilon * f - af\|_{L^\infty(K)} \to 0$ as $\varepsilon \to 0$.