4.1.8. The beta function is defined in Appendix A.2. Derive the identity

$$t^{\alpha} = \frac{1}{B(\alpha-\beta,\beta+1)} \int_0^t (t-s)^{\alpha-\beta-1} s^{\beta}\, ds$$

and show that the function $K_R^{\alpha}(x) = \sum_{|m|\le R} \left(1-\frac{|m|^2}{R^2}\right)^{\alpha} e^{2\pi i m\cdot x}$ satisfies (4.1.17).
$\big[$*Hint:* Take $t = 1-\frac{|m|^2}{R^2}$ and change variables $s = \frac{r^2-|m|^2}{R^2}$ in the displayed identity.$\big]$

4.2 A. E. Divergence of Fourier Series and Bochner–Riesz means

We saw in Proposition 3.4.6 that the Fourier series of a continuous function may diverge at a point. As expected, the situation can only get worse as the functions get worse. In this section we present an example, due to A. N. Kolmogorov, of an integrable function on $\mathbf{T}^1$ whose Fourier series diverges almost everywhere. We also prove an analogous result for the Bochner–Riesz means at the critical index.

4.2.1 Divergence of Fourier Series of Integrable Functions

It is natural to start our investigation with the case $n = 1$. We begin with the following important result:

Theorem 4.2.1. *There exists an integrable function on the circle* $\mathbf{T}^1$ *whose Fourier series diverges almost everywhere.*

Proof. The proof of this theorem is a bit involved, and we need a sequence of lemmas, which we prove first.

Lemma 4.2.2. ***(Kronecker)*** *Suppose that* $N \in \mathbf{Z}^+$ *and*

$$\{x_1, x_2, \dots, x_N, 1\}$$

is a linearly independent set over the rationals. Then for any $\varepsilon > 0$ *and any complex numbers* $z_1, z_2, \dots, z_N$ *with* $|z_j| = 1$*, there exists an integer* $L \in \mathbf{Z}$ *such that*

$$|e^{2\pi i L x_j} - z_j| < \varepsilon \qquad \text{for all} \quad 1 \le j \le N.$$

Proof. Suppose that the assertion claimed is false. Then there is an $\varepsilon > 0$ and complex numbers $z_j = e^{2\pi i\theta_j}$, $j = 1, \dots, N$, with $0 \le \theta_j < 1$, such that

$$\big\{mx_1 (\text{mod } 1), \dots, mx_N (\text{mod } 1)) : \ m \in \mathbf{Z}\big\} \cap B\big((\theta_1, \dots, \theta_N), \varepsilon\big) = \emptyset,$$