for all $x \in \mathbf{T}^n$. We show that the family $\{L^{z,R}\}_{R>0}$ is an approximate identity on $\mathbf{T}^n$ when $\mathrm{Re}\, z > \frac{n-1}{2}$; on this see the related Exercise 3.1.3. Obviously, using (4.1.19) we have that

$$\int_{\mathbf{T}^n} |L^{z,R}(x)|\,dx = \int_{\mathbf{R}^n} |K^z(y)|\,dy = C''(n, \mathrm{Re}\, z) e^{10\,|\mathrm{Im}\, z|^2} < \infty \tag{4.1.20}$$

for some constant $C''(n, \mathrm{Re}\, z)$, and also

$$\int_{\mathbf{T}^n} L^{z,R}(x)\,dx = \int_{\mathbf{R}^n} K^z(y)\,dy = m_z(0) = 1$$

for all $R > 0$ when $\mathrm{Re}\, z > \frac{n-1}{2}$. Moreover, for $\delta < \frac{1}{2}$ using (4.1.19) we have

$$\int_{\delta \le \sup_j |x_j| \le \frac{1}{2}} \big|L^{z,R}(x)\big|\,dx \le \frac{C_{n,z}}{R^{\mathrm{Re}\, z - \frac{n-1}{2}}} \int_{\delta \le \sup_j |x_j| \le \frac{1}{2}} \sum_{\ell \in \mathbf{Z}^n} \frac{1}{|x+\ell|^{n+\mathrm{Re}\, z - \frac{n-1}{2}}}\,dx \to 0\,,$$

thus the integral of $L^{z,R}$ over $[-1/2,1/2]^n \setminus [-\delta,\delta]^n$ tends to zero as $R \to \infty$.

Using Theorem 1.2.19, we obtain these conclusions for $\mathrm{Re}\, z > \frac{n-1}{2}$:

(a) For $f \in L^1(\mathbf{T}^n)$, $B_R^z(f)$ converge to $f$ in $L^1$ as $R \to \infty$.
(b) For $f$ continuous on $\mathbf{T}^n$, $B_R^z(f)$ converge to $f$ uniformly as $R \to \infty$.

We turn to the corresponding results for $1 < p < \infty$. We have that

$$\mathrm{Re}\, z > \tfrac{n-1}{2} \implies \sup_{R>0} \big\|B_R^z\big\|_{L^1(\mathbf{T}^n)\to L^1(\mathbf{T}^n)} = C''(n, \mathrm{Re}\, z) e^{10\,|\mathrm{Im}\, z|^2} \tag{4.1.21}$$

$$\mathrm{Re}\, z = 0 \implies \sup_{R>0} \big\|B_R^z\big\|_{L^2(\mathbf{T}^n)\to L^2(\mathbf{T}^n)} = \|m_z\|_{L^\infty} = 1\,. \tag{4.1.22}$$

The family of operators $f \mapsto B_R^z(f)$ is of admissible growth for all $\mathrm{Re}\, z \ge 0$, since for all measurable subsets $A$, $B$ of $\mathbf{T}^n$ we have

$$\bigg|\int_{\mathbf{T}^n} B_R^z(\chi_A)\chi_B\,dx\bigg| = \bigg|\sum_{k\in\mathbf{Z}^n} \widehat{\chi_A}(k/R) m^z(k) \overline{\widehat{\chi_B}(k)}\bigg| \le \sum_{|k|\le R} 1 \le C_n R^n\,,$$

thus condition (1.3.23) holds. Moreover, hypothesis (1.3.24) of Theorem 1.3.7 holds in view of (4.1.21) and (4.1.22). Applying Theorem 1.3.7 (or rather Exercise 1.3.4 in which the strip $[0,1] \times \mathbf{R}$ is replaced by the more general strip $[a,b] \times \mathbf{R}$) we obtain that when $\alpha = \mathrm{Re}\, z > (n-1)|\frac{1}{p} - \frac{1}{2}|$, we have

$$\sup_{R>0} \big\|B_R^\alpha\big\|_{L^p(\mathbf{T}^n)\to L^p(\mathbf{T}^n)} < \infty\,.$$

Finally, using Corollary 4.1.3, we deduce that $B_R^\alpha(f) \to f$ in $L^p(\mathbf{T}^n)$ as $R \to \infty$ for all $f \in L^p(\mathbf{T}^n)$. □

The preceding result is sharp in the case $p = 1$ (Theorem 4.2.5). For this reason, the number $\alpha = (n-1)/2$ is referred to as the *critical index of Bochner–Riesz summability*.