$$S'_N(g)(x) = \sum_{m=0}^{2N} \widehat{g}(m) e^{2\pi i m x}$$

from L^p to L^p. Therefore,

$$\sup_{N\geq 0} \big\| S_N \big\|_{L^p \to L^p} < \infty \iff \sup_{N\geq 0} \big\| S'_N \big\|_{L^p \to L^p} < \infty, \tag{4.1.10}$$

and both of these statements are equivalent to the fact that for all $f \in L^p(\mathbf{T}^1)$, $S_N(f) \to f$ in L^p as $N \to \infty$.

We have already observed that the L^p boundedness of the conjugate function is equivalent to that of P_+. Therefore, it suffices to show that the L^p boundedness of P_+ is equivalent to the uniform L^p boundedness of S'_N.

Suppose first that $\sup_{N\geq 0} \|S'_N\|_{L^p \to L^p} < \infty$. Theorem 4.1.1 applied to the sequence $a(m,R) = 1$ for $0 \leq m \leq R$ and $a(m,R) = 0$ otherwise gives that the operator $A(f) = P_+(f) + \widehat{f}(0)$ is bounded on $L^p(\mathbf{T}^1)$. Hence so is P_+.

Conversely, suppose that P_+ extends to a bounded operator from $L^p(\mathbf{T}^1)$ to itself. For all h in $\mathscr{C}^\infty(\mathbf{T}^1)$ we can write

$$\begin{aligned} S'_N(h)(x) &= \sum_{m=0}^{\infty} \widehat{h}(m) e^{2\pi i m x} - \sum_{m=2N+1}^{\infty} \widehat{h}(m) e^{2\pi i m x} \\ &= \sum_{m=1}^{\infty} \widehat{h}(m) e^{2\pi i m x} + \widehat{h}(0) - e^{2\pi i (2N) x} \sum_{m=1}^{\infty} \widehat{h}(m+2N) e^{2\pi i m x} \\ &= P_+(h)(x) - e^{2\pi i (2N) x} P_+(e^{-2\pi i (2N)(\cdot)} h)(x) + \widehat{h}(0). \end{aligned}$$

This identity implies that

$$\sup_{N\geq 0} \big\| S'_N(f) \big\|_{L^p} \leq \big(2\big\| P_+ \big\|_{L^p \to L^p} + 1\big) \big\| f \big\|_{L^p} \tag{4.1.11}$$

for all f smooth, and by density for all $f \in L^p(\mathbf{T}^1)$. Note that S'_N is well defined on $L^p(\mathbf{T}^1)$. Thus the operators S'_N are uniformly bounded on $L^p(\mathbf{T}^n)$.

Thus the uniform L^p boundedness of S_N is equivalent to the uniform L^p boundedness of S'_N, which is equivalent to the L^p boundedness of P_+, which in turn is equivalent to the L^p boundedness of the conjugate function. □

4.1.2 The L^p Boundedness of the Conjugate Function

We know now that convergence of Fourier series in L^p is equivalent to the L^p boundedness of the conjugate function or either of the two Riesz projections. It is natural to ask whether these operators are L^p bounded.