is a bounded linear functional on $\mathscr{C}_E$ with norm at most K. By the Hanh-Banach theorem this functional admits an extension to $\mathscr{C}(\mathbf{T}^1)$ with the same norm. Hence there is a measure μ, whose total variation $\|\mu\|_{\mathscr{M}}$ does not exceed K, such that

$$\sum_{m\in E} \widehat{f}(m)b(m) = \int_{\mathbf{T}^1} f(t)\,d\mu(t)\,.$$

Taking $f(t) = e^{2\pi imt}$ in (2) we obtain $\widehat{\mu}(m) = b(m)$ for all $m \in E$.

If (4) holds and $b(m) \to 0$ as $|m| \to \infty$, using Lemma 3.3.2 there is a convex sequence $c(m)$ such that $c(m) > 0$, $c(m) \to 0$ as $|m| \to \infty$, $c(-m) = c(m)$, and $|b(m)| \le c(m)$ for all $m \in \mathbf{Z}$. By (4), there is a finite Borel measure μ with $\widehat{\mu}(m) = b(m)/c(m)$ for all $m \in E$.

By Theorem 3.3.4, there is a function g in $L^1(\mathbf{T}^1)$ such that $\widehat{g}(m) = c(m)$ for all $m \in \mathbf{Z}$. Then $b(m) = \widehat{g}(m)\widehat{\mu}(m)$ for all $m \in E$. Since $f = g * \mu$ is in L^1, we have $b(m) = \widehat{f}(m)$ for all $m \in E$, and thus (4) implies (5).

Finally, if (5) holds, we show (3). Given $f \in \mathscr{C}_E$, we show that for an arbitrary sequence d_m tending to zero, we have $\sum_{m\in\mathbf{Z}} |\widehat{f}(m)d_m| < \infty$; this implies that $\sum_{m\in\mathbf{Z}} |\widehat{f}(m)| < \infty$. Given a sequence $d_m \to 0$, pick a function g in L^1 such that $\widehat{g}(m)\widehat{f}(m) = |\widehat{f}(m)|\,|d_m|$ for all $m \in E$ by assumption (5). Then the series

$$\sum_{m\in\mathbf{Z}} \widehat{g}(m)\widehat{f}(m) = \sum_{m\in\mathbf{Z}} \widehat{f*g}(m) \tag{3.6.25}$$

has nonnegative terms and the function $f * g$ is continuous, thus $F_N * (f * g)(0) \to (f * g)(0)$ as $N \to \infty$. It follows that $D_N * (f * g)(0) \to (f * g)(0)$, thus the series in (3.6.25) converges (see Exercise 3.5.4) and hence $\sum_{m\in\mathbf{Z}} |\widehat{f}(m)d_m| < \infty$. $\square$

Example 3.6.10. Every lacunary set is a Sidon set. Indeed, suppose that E is a lacunary set with constant A. If f is a continuous function which satisfies (3.6.24), then Theorem 3.6.6 gives that

$$\sum_{m\in\Lambda} |\widehat{f}(m)| \le C(A)\big\|f\big\|_{L^\infty} < \infty;$$

hence f has an absolutely convergent Fourier series.

Example 3.6.11. There exist subsets of $\mathbf{Z}$ that are not Sidon. For example, $\mathbf{Z}\setminus\{0\}$ is not a Sidon set. See Exercise 3.6.7.

Exercises

3.6.1. Suppose that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$ is a lacunary sequence of integers with constant $A \ge 3$. Prove that for every integer m there exists at most one N-tuple $(\varepsilon_1, \dots, \varepsilon_N)$ with each $\varepsilon_j \in \{-1, 1, 0\}$ such that

$$m = \varepsilon_1\lambda_1 + \cdots + \varepsilon_N\lambda_N\,.$$