it would follow that $A^r \le m$, which contradicts our choice of r. Likewise, we eliminate the case $\max(\mu_{j_1},\dots,\mu_{j_m}) < \max(\mu_{k_1},\dots,\mu_{k_m})$. We conclude that these numbers are equal. We can now continue the same reasoning using induction to conclude that if $\mu_{j_1}+\cdots+\mu_{j_m} = \mu_{k_1}+\cdots+\mu_{k_m}$, then

$$\{\mu_{k_1},\dots,\mu_{k_m}\} = \{\mu_{j_1},\dots,\mu_{j_m}\}.$$

In view of this fact and of the permutations involved in the previous sum, we obtain

$$\int_0^1 |\varphi_s(x)|^{2m}\,dx \le m! \sum_{j_1=1}^{N}\cdots\sum_{j_m=1}^{N} |\widehat{\varphi_s}(\mu_{j_1})|^2\cdots|\widehat{\varphi_s}(\mu_{j_m})|^2 = m!\big(\|\varphi_s\|_{L^2}^2\big)^m,$$

which implies that $\|\varphi_s\|_{L^{2m}} \le (m!)^{\frac{1}{2m}}\|\varphi_s\|_{L^2}$ for all $s\in\{1,2,\dots,r\}$. Then we have

$$\|f_N\|_{L^p} \le \|f_N\|_{L^{2m}} \le \sqrt{r}\Big(\sum_{s=1}^{r}\|\varphi_s\|_{L^{2m}}^2\Big)^{\frac12} \le c_{r,m}\Big(\sum_{s=1}^{r}\|\varphi_s\|_{L^2}^2\Big)^{\frac12} = c_{r,m}\|f_N\|_{L^2},$$

since the functions φ_s are orthogonal in L^2. Here $\sqrt{r}\,(m!)^{\frac{1}{2m}}$. Since r can be chosen to be $[\log_A m]+1$ and m can be taken to be $[\frac{p}{2}]+1$, for every f_N of the form (3.6.10), we have now established the inequality

$$\|f_N\|_{L^p(\mathbf{T}^1)} \le c_p(A)\|f_N\|_{L^2(\mathbf{T}^1)}, \qquad p\ge 2, \tag{3.6.11}$$

with $c_p(A) = \sqrt{1+\big[\log_A\big([\frac{p}{2}]+1\big)\big]}\,\big(([\frac{p}{2}]+1)!\big)^{\frac{1}{[\frac{p}{2}]+1}}$.

To replace f_N by f in (3.6.11), we recall our assumption that $f\in L^2(\mathbf{T}^1)$. We observe that $f_N\to f$ in L^2 and thus f_{N_j} tends to f a.e. for some subsequence. Then Fatou's lemma and (3.6.11) imply for $1<p<\infty$

$$\begin{aligned}\int_0^1 |f(x)|^p\,dx &= \int_0^1 \liminf_{j\to\infty}|f_{N_j}(x)|^p\,dx\\ &\le \liminf_{j\to\infty}\int_0^1 |f_{N_j}(x)|^p\,dx\\ &\le c_p(A)^p\liminf_{j\to\infty}\|f_{N_j}\|_{L^2}^p\\ &= c_p(A)^p\|f\|_{L^2}^p.\end{aligned}$$

We conclude that

$$\|f\|_{L^p(\mathbf{T}^1)} \le c_p(A)\|f\|_{L^2(\mathbf{T}^1)}, \qquad p\ge 2. \tag{3.6.12}$$

By interpolation we obtain

$$\|f\|_{L^2} \le \|f\|_{L^4}^{\frac23}\|f\|_{L^1}^{\frac13} \le \big([\log_A 3]+1\big)^{\frac12\cdot\frac23}\|f\|_{L^2}^{\frac23}\|f\|_{L^1}^{\frac13}.$$