Using (3.6.1) and (3.6.2), we obtain that for any $m \in \mathbf{Z}$ we have

$$1 \le |m - \lambda_k| < (1 - A^{-1})\lambda_k \implies \widehat{f}(m) = 0. \tag{3.6.3}$$

Let $[t]$ denote the integer part of t. Given $\varepsilon > 0$, pick a positive integer k_0 such that if $[(1 - A^{-1})\lambda_{k_0}] = 2N_0$, then $N_0^{-2} < \varepsilon$, and

$$\sup_{|x| < N_0^{-\frac{1}{4}}} \left| \frac{f(x)}{x} \right| < \varepsilon. \tag{3.6.4}$$

The expression in (3.6.4) can be made arbitrarily small, since f is differentiable at the origin. Now take an integer k with $k \ge k_0$ and set $2N = [\min(A-1, 1-A^{-1})\lambda_k] = [(A-1)\lambda_k]$, which is of course at least $2N_0$. Using (3.6.3), we obtain that for any trigonometric polynomial K_N of degree $2N$ with $\widehat{K_N}(0) = 1$ we have

$$\widehat{f}(\lambda_k) = \int_{|x| \le \frac{1}{2}} f(x) K_N(x) e^{-2\pi i \lambda_k x}\, dx. \tag{3.6.5}$$

We take $K_N = (F_N / \|F_N\|_{L^2})^2$, where F_N is the Fejér kernel. Using (3.1.18), we obtain first the identity

$$\|F_N\|_{L^2}^2 = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right)^2 = 1 + \frac{1}{3}\frac{N(2N+1)}{N+1} > \frac{N}{3} \tag{3.6.6}$$

and also the estimate

$$F_N(x)^2 \le \left(\frac{1}{N+1}\frac{1}{4x^2}\right)^2, \tag{3.6.7}$$

which is valid for $|x| \le 1/2$. In view of (3.6.6) and (3.6.7), we have the estimate

$$K_N(x) \le \frac{3}{16}\frac{1}{N^3}\frac{1}{x^4}. \tag{3.6.8}$$

We now use (3.6.5) to obtain

$$\lambda_k \widehat{f}(\lambda_k) = \lambda_k \int_{|x| \le \frac{1}{2}} f(x) K_N(x) e^{-2\pi i \lambda_k x}\, dx = I_k^1 + I_k^2 + I_k^3,$$

where

$$\begin{aligned} I_k^1 &= \lambda_k \int_{|x| \le N^{-1}} f(x) K_N(x) e^{-2\pi i \lambda_k x}\, dx, \\ I_k^2 &= \lambda_k \int_{N^{-1} < |x| \le N^{-\frac{1}{4}}} f(x) K_N(x) e^{-2\pi i \lambda_k x}\, dx, \\ I_k^3 &= \lambda_k \int_{N^{-\frac{1}{4}} < |x| \le \frac{1}{2}} f(x) K_N(x) e^{-2\pi i \lambda_k x}\, dx. \end{aligned}$$