3.5.3. Let $L^1_1(\mathbf{T}^1)$ be the space of all differentiable functions on $\mathbf{T}^1$ whose derivatives are integrable. Obtain the inclusions $L^1_1(\mathbf{T}^1) \subseteq BV(\mathbf{T}^1) \subseteq L^\infty(\mathbf{T}^1)$ as follows:
(a) If $f \in L^1_1(\mathbf{T}^1)$, then $\mathrm{Var}(f) \le \|f'\|_{L^1}$.
(b) If $f \in BV(\mathbf{T}^1)$, then $\|f\|_{L^\infty} \le \mathrm{Var}(f) + |f(0)|$.

3.5.4. (a) Let $a_k \ge 0$, $s_N = \sum_{k=-N}^N a_k$, and $\sigma_N = \frac{1}{N+1}(s_0 + \cdots + s_N)$. Suppose that $\sigma_N \to L < \infty$ as $N \to \infty$. Prove that $s_N \to L$ as $N \to \infty$.
(b) Apply the preceding result to show that if a complex-valued function h on $\mathbf{T}^1$ is continuous in a neighborhood of 0 and $\widehat{h}(m) \ge 0$ for all $m \in \mathbf{Z}$, then $h(0) \ge 0$ and $\sum_{m\in\mathbf{Z}} \widehat{h}(m) = h(0) < \infty$; i.e., the partial sums of the Fourier series of h converge at zero.

3.5.5. Let $h \in L^1(\mathbf{T}^1)$, $t_0 \in \mathbf{T}^1$, and $0 < \delta < 1/2$.
(a) Show that $(h * D_N)(t_0) \to L$ as $N \to \infty$ if and only if

$$\lim_{M\to\infty} \int_0^\delta \left(\frac{h(t_0-t)+h(t_0+t)}{2} - L \right) \frac{\sin(Mt)}{t}\, dt = 0 .$$

(b) Conclude that if an integrable function h on $\mathbf{T}^1$ satisfies

$$\int_0^\delta \frac{|h(t_0-t)+h(t_0+t)-2L|}{t}\, dt < \infty ,$$

then $(h * D_N)(t_0) \to L$ as $N \to \infty$.
(c) In particular, if there are constants $C, \beta > 0$ with $\beta < 1$ such that for all t with $0 < t < \delta$ we have

$$|h(t_0-t)+h(t_0+t)-2h(t_0)| \le C t^\beta ,$$

then $(h * D_N)(t_0) \to h(t_0)$ as $N \to \infty$.
(d) If h is an odd function, then $(h * D_N)(0) \to 0$ as $N \to \infty$.

3.5.6. Let $f \in L^1(\mathbf{T}^1)$ and suppose that (a,b) is an interval in $\mathbf{T}^1$. Then we have

$$\lim_{N\to\infty} \int_a^b (f * D_N)(t)\, dt = \int_a^b f(t)\, dt .$$

$\big[$*Hint:* Use Theorems 3.5.4 and 3.5.5 and the fact that the operator $f \mapsto f * D_N$ is self-adjoint.$\big]$

3.6 Lacunary Series and Sidon Sets

Lacunary series provide examples of 1-periodic functions on the line that possess certain remarkable properties.