### 3.4.4 Pointwise Convergence of the Dirichlet Means

We have seen that continuous functions may have divergent Fourier series. How about Lipschitz continuous functions? As it turns out, there is a more general condition that implies convergence for the Fourier series of functions that satisfy a certain integrability condition.

**Theorem 3.4.7.** ***(Dini)*** *Let $f$ be an integrable function on $\mathbf{T}^1$, let $t_0$ be a point on $\mathbf{T}^1$ for which $f(t_0)$ is defined and assume that*

$$\int_{|t|\le\frac{1}{2}} \frac{|f(t+t_0)-f(t_0)|}{|t|}\,dt < \infty. \tag{3.4.9}$$

*Then $(D_N * f)(t_0) \to f(t_0)$ as $N\to\infty$.*
***(Tonelli)*** *Let $f$ be an integrable function on $\mathbf{T}^n$ and let $a=(a_1,\dots,a_n)\in\mathbf{T}^n$. If $f$ is defined at $a$ and*

$$\int_{|x_1|\le\frac{1}{2}} \cdots \int_{|x_n|\le\frac{1}{2}} \frac{|f(x+a)-f(a)|}{|x_1|\cdots|x_n|}\,dx_n\cdots dx_1 < \infty, \tag{3.4.10}$$

*then we have $(D_N^n * f)(a)\to f(a)$ as $N\to\infty$.*

*Proof.* Since the one-dimensional result is contained in the multidimensional one, we prove the latter. Replacing $f(x)$ by $f(x+a)-f(a)$, we may assume that $a=0$ and $f(a)=0$. Using identities (3.1.15) and (3.1.14), we can write

$$\begin{aligned}(D_N^n * f)(0) &= \int_{\mathbf{T}^n} f(-x)\prod_{j=1}^n \frac{\sin((2N+1)\pi x_j)}{\sin(\pi x_j)}\,dx_n\cdots dx_1 \qquad (3.4.11)\\ &= \int_{\mathbf{T}^n} f(-x)\prod_{j=1}^n \left(\frac{\sin(2N\pi x_j)\cos(\pi x_j)}{\sin(\pi x_j)} + \cos(2N\pi x_j)\right)dx_n\cdots dx_1\,.\end{aligned}$$

Expand out the product to express the integrand as a sum of terms of the form

$$\left\{ f(-x)\prod_{j\in I}\frac{\cos(\pi x_j)}{\sin(\pi x_j)}\right\}\prod_{j\in I}\sin(2N\pi x_j)\prod_{k\in\{1,2,\dots,n\}\setminus I}\cos(2N\pi x_k)\,, \tag{3.4.12}$$

where $I$ is a subset of $\{1,2,\dots,n\}$; here we use the convention that the product over an empty set of indices is 1. The function $f_I$ inside the curly brackets in (3.4.12) is integrable on $[-\frac{1}{2},\frac{1}{2})^n$ except possibly in a neighborhood of the origin, since $|\sin(\pi x_j)|\ge 2|x_j|$ when $|x_j|\le\frac{1}{2}$. But condition (3.4.10) with $a=0$ and $f(a)=0$ guarantees that $f_I$ is also integrable in a neighborhood of the origin. Expressing the sines and cosines in (3.4.12) in terms of exponentials, we obtain that the integral of (3.4.12) over $[-\frac{1}{2},\frac{1}{2})^n$ is a finite linear combination of Fourier coefficients of $f_I$ at the points $(\pm N,\dots,\pm N)\in\mathbf{Z}^n$. Applying Lemma 3.3.1 yields that the expression in (3.4.11) tends to zero as $N\to\infty$. □