$$\begin{aligned}(g*g)(x) &= \big|B(0,1)\cap B(x,1)\big| = \int_{-\sqrt{1-\frac{1}{4}|x|^2}}^{+\sqrt{1-\frac{1}{4}|x|^2}} \Big(2\sqrt{1-t^2}-|x|\Big)\,dt\\ &= 2\arcsin\left(\sqrt{1-\tfrac{1}{4}|x|^2}\right) - |x|\sqrt{1-\tfrac{1}{4}|x|^2}\end{aligned}$$

when $x=(x_1,x_2)$ in $\mathbf{R}^2$ satisfies $|x|\le 2$, while $(g*g)(x)=0$ if $|x|\ge 2$.

A calculation similar to that in Remark 1.2.7 yields that

$$\big\|f*g\big\|_{L^1(G)} \le \big\|f\big\|_{L^1(G)}\big\|g\big\|_{L^1(G)}\,, \tag{1.2.6}$$

that is, the convolution of two integrable functions is also an integrable function with L^1 norm less than or equal to the product of the L^1 norms.

Proposition 1.2.9. *For all f, g, h in $L^1(G)$, the following properties are valid:*

(1) $f*(g*h)=(f*g)*h$ (associativity)
(2) $f*(g+h)=f*g+f*h$ and $(f+g)*h=f*h+g*h$ (distributivity)

Proof. The easy proofs are omitted. □

Proposition 1.2.9 and (1.2.6) imply that $L^1(G)$ is a (not necessarily commutative) Banach algebra under the convolution product.

1.2.3 Basic Convolution Inequalities

The most fundamental inequality involving convolutions is the following.

Theorem 1.2.10. ***(Minkowski's inequality)*** *Let $1\le p\le\infty$. For f in $L^p(G)$ and g in $L^1(G)$ we have that $g*f$ exists λ-a.e. and satisfies*

$$\big\|g*f\big\|_{L^p(G)} \le \big\|g\big\|_{L^1(G)}\big\|f\big\|_{L^p(G)}\,. \tag{1.2.7}$$

Proof. Estimate (1.2.7) follows directly from Exercise 1.1.6. Here we give a direct proof. We may assume that $1<p<\infty$, since the cases $p=1$ and $p=\infty$ are simple. We first show that the convolution $|g|*|f|$ exists λ-a.e. Indeed,

$$(|g|*|f|)(x) = \int_G |f(y^{-1}x)|\,|g(y)|\,d\lambda(y)\,. \tag{1.2.8}$$

Apply Hölder's inequality in (1.2.8) with respect to the measure $|g(y)|\,d\lambda(y)$ to the functions $y\mapsto f(y^{-1}x)$ and 1 with exponents p and $p'=p/(p-1)$, respectively. We obtain

$$(|g|*|f|)(x) \le \left(\int_G |f(y^{-1}x)|^p|g(y)|\,d\lambda(y)\right)^{\frac{1}{p}}\left(\int_G |g(y)|\,d\lambda(y)\right)^{\frac{1}{p'}}\,. \tag{1.2.9}$$