Moreover, the L^p conclusion about $\mathscr{H}$ follows from the weak type $(1,1)$ result and the trivial L^∞ inequality, in view of the Marcinkiewicz interpolation theorem (Theorem 1.3.2). The required weak type $(1,1)$ estimate for $\mathscr{G}$ on $\mathbf{R}^n$ is a consequence of Lemma 3.4.5. Modulo the proof of this lemma, part (a) of the theorem is proved.

To prove the statement in part (b) observe that for $f \in \mathscr{C}^\infty(\mathbf{T}^n)$, which is a dense subspace of L^1, we have $F_N^n * f \to f$ uniformly on $\mathbf{T}^n$ as $N \to \infty$, since the sequence $\{F_N\}_N$ is an approximate identity. Since by part (a), $\mathscr{H}$ maps $L^1(\mathbf{T}^n)$ to $L^{1,\infty}(\mathbf{T}^n)$, Theorem 2.1.14 yields that for $f \in L^1(\mathbf{T}^n)$, $F_N^n * f \to f$ a.e. $\square$

We now prove the weak type $(1,1)$ boundedness of $\mathscr{G}$ used earlier.

Lemma 3.4.5. *Let $\Phi(x_1,\dots,x_n) = (1+|x_1|^2)^{-1}\cdots(1+|x_n|^2)^{-1}$ and for $\varepsilon > 0$ let $\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(\varepsilon^{-1}x)$. Then the maximal operator*

$$\mathscr{G}(f) = \sup_{\varepsilon>0} |f| * \Phi_\varepsilon$$

maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$.

Proof. Let $I_0 = [-1,1]$ and $I_k = \{t \in \mathbf{R} : 2^{k-1} \le |t| \le 2^k\}$ for $k = 1,2,\dots$. Also, let $\widetilde{I_k}$ be the convex hull of I_k, that is, the interval $[-2^k, 2^k]$. For $a_2,\dots,a_n$ fixed positive numbers, let $M_{a_2,\dots,a_n}$ be the maximal operator obtained by averaging a function on $\mathbf{R}^n$ over all products of closed intervals $J_1 \times \cdots \times J_n$ containing a given point with

$$|J_1| = 2^{a_2}|J_2| = \cdots = 2^{a_n}|J_n|.$$

In view of Exercise 2.1.9(c), we have that $M_{a_2,\dots,a_n}$ maps L^1 to $L^{1,\infty}$ with some constant independent of the a_j's. (This is due to the nice doubling property of this family of rectangles.) For a fixed $\varepsilon > 0$ we estimate the expression

$$(\Phi_\varepsilon * |f|)(0) = \int_{\mathbf{R}^n} \frac{|f(-\varepsilon y)|\,dy}{(1+y_1^2)\cdots(1+y_n^2)}\,.$$

Split $\mathbf{R}^n$ into $n!$ regions of the form $|y_{j_1}| \ge \cdots \ge |y_{j_n}|$, where $\{j_1,\dots,j_n\}$ is a permutation of the set $\{1,\dots,n\}$ and $y = (y_1,\dots,y_n)$. By symmetry, we examine the region where $|y_1| \ge \cdots \ge |y_n|$ which we split in finitely many regions $\mathscr{R}$ of the form $|y_1| \ge \cdots \ge |y_\ell| \ge 1 \ge |y_{\ell+1}| \ge |y_n|$. Then for some constant $C > 0$ we have

$$\int_{\mathscr{R}} \frac{|f(-\varepsilon y)|\,dy}{(1+y_1^2)\cdots(1+y_n^2)} \le C \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_n=0}^{k_{\ell-1}} 2^{-(2k_1+\cdots+2k_\ell)} \int\limits_{I_{k_1}\times\cdots\times I_{k_\ell}\times I_0^{n-\ell}} |f(-\varepsilon y)|\,dy_n\cdots dy_1,$$

and the last expression is trivially controlled by the corresponding expression, where the I_k's are replaced by the $\widetilde{I_k}$'s. This, in turn, is controlled by

$$C' \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\ell=0}^{k_{\ell-1}} 2^{-(k_1+\cdots+k_\ell)} M_{k_1-k_2,\dots,k_1-k_\ell,0,\dots,0}(f)(0)\,. \tag{3.4.8}$$

Now set $s_2 = k_1 - k_2, \dots, s_\ell = k_1 - k_\ell$, observe that $s_j \geq 0$, use that

$$2^{-(k_1+\cdots+k_\ell)} \leq 2^{-\frac{k_1}{2}} 2^{-\frac{s_2}{2\ell}} \cdots 2^{-\frac{s_\ell}{2\ell}},$$

and change the indices of summation to estimate the expression in (3.4.8) by

$$C'' \sum_{k_1=0}^{\infty} \sum_{s_2=0}^{\infty} \cdots \sum_{s_\ell=0}^{\infty} 2^{-\frac{k_1}{2}} 2^{-\frac{s_2}{2\ell}} \cdots 2^{-\frac{s_\ell}{2\ell}} M_{s_2,\dots,s_\ell,0,\dots,0}(f)(0).$$

Argue similarly for other ℓ and all remaining regions $|y_{j_1}| \geq \cdots \geq |y_{j_n}|$. Finally, translate to an arbitrary point x and take the supremum over $\varepsilon > 0$ to obtain that $\mathscr{G}(f)(x)$ is bounded by a finite sum of expressions of the form

$$C'' \sum_{s_2=0}^{\infty} \cdots \sum_{s_\ell=0}^{\infty} 2^{-\frac{s_2}{2n}} \cdots 2^{-\frac{s_\ell}{2n}} M_{s'_2,\dots,s'_n}(f)(x)$$

where $\ell \leq n$ and $(s'_2,\dots,s'_n)$ ranges over all permutations of $(s_2,\dots,s_\ell,0,\dots,0)$. Now use the fact that the maximal functions $M_{s'_2,\dots,s'_n}$ map L^1 to $L^{1,\infty}$ uniformly in $s'_2,\dots,s'_n$ and the result of Exercise 1.4.10 to obtain the desired conclusion for $\mathscr{G}$. $\square$

3.4.3 Pointwise Divergence of the Dirichlet Means

We now pass to the more difficult question of convergence of the square partial sums of a Fourier series. It is natural to start our investigation with the class of continuous functions. Do the partial sums of the Fourier series of continuous functions converge pointwise? The following simple proposition warns about the behavior of partial sums.

Proposition 3.4.6. *(a) (**duBois Reymond**) There exists a continuous function f on $\mathbf{T}^1$ whose partial sums diverge at a point. Precisely, for some point $x_0 \in \mathbf{T}^1$ we have*

$$\limsup_{N\to\infty} \Bigg| \sum_{\substack{m\in\mathbf{Z} \\ |m_j|\leq N}} \widehat{f}(m) e^{2\pi i x_0 m} \Bigg| = \infty.$$

(b) There exists a continuous function F on $\mathbf{T}^n$ and $x_0 \in \mathbf{T}^1$ such that the sequence

$$\limsup_{N\to\infty} \Bigg| \sum_{\substack{m\in\mathbf{Z}^n \\ |m_j|\leq N}} \widehat{F}(m) e^{2\pi i (x_0 m_1 + x_2 m_2 + \cdots + x_n m_n)} \Bigg| = \infty$$

for all $x_2,\dots,x_n$ in $\mathbf{T}^1$.