$$\sum_{j=0}^{N} e^{2\pi i jx} = \frac{e^{2\pi i(N+1)x}-1}{e^{2\pi ix}-1} = \frac{e^{\pi i(N+1)x}}{e^{\pi ix}} \frac{e^{\pi i(N+1)x}-e^{-\pi i(N+1)x}}{e^{\pi ix}-e^{-\pi ix}}$$

from which it follows that

$$\sum_{j=0}^{N} e^{2\pi i jx} = e^{\pi i N x} \frac{\sin(\pi(N+1)x)}{\sin(\pi x)}. \tag{3.1.19}$$

Taking absolute values and squaring both sides in (3.1.19) we obtain

$$\left| \frac{\sin(\pi(N+1)x)}{\sin(\pi x)} \right|^2 = \left| \sum_{j=0}^{N} e^{2\pi i jx} \right|^2. \tag{3.1.20}$$

Expanding the square on the right in (3.1.20) yields the identity

$$\frac{1}{N+1} \frac{\sin^2(\pi(N+1)x)}{\sin^2(\pi x)} = \frac{1}{N+1} \sum_{k=0}^{N} \sum_{l=0}^{N} e^{2\pi i(k-l)x}. \tag{3.1.21}$$

The number of times the index $k-l$ in the double sum in (3.1.21) takes the value $j \in \{-N, -(N-1), \dots, 0, \dots, (N-1), N\}$ is actually $N+1-|j|$ and is shown in the following table:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & 1 & 2 & \cdots & N-2 & N-1 & N \\
 & & & & & -1 & 0 & 1 & 2 & \cdots & N-2 & N-1 & \\
 & & & & -2 & -1 & 0 & 1 & 2 & \cdots & N-2 & & \\
 & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\
 & & -(N-2) & \cdots & -2 & -1 & 0 & 1 & 2 & & & & \\
 & -(N-1) & -(N-2) & \cdots & -2 & -1 & 0 & 1 & & & & & \\
-N & -(N-1) & -(N-2) & \cdots & -2 & -1 & 0 & & & & & &
\end{array}$$

We obtain that

$$\frac{1}{N+1} \sum_{k=0}^{N} \sum_{l=0}^{N} e^{2\pi i(k-l)x} = \frac{1}{N+1} \sum_{j=-N}^{N} (N+1-|j|) e^{2\pi i jx},$$

and this combined with (3.1.21) yields

$$\sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) e^{2\pi i jx} = \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)} \right)^2. \tag{3.1.22}$$

This proves the second identity in (3.1.18). □

Definition 3.1.8. Let N be a nonnegative integer. The function F_N on $\mathbf{T}^1$ given by (3.1.22) is called the *Fejér kernel*.

The Fejér kernel F_N^n on $\mathbf{T}^n$ is defined as the product of the 1-dimensional Fejér kernels, or as the average of the product of the Dirichlet kernels in each variable, precisely, $F_N^1(x) = F_N(x)$ and