$\mathbf{T}^n = [0,1]^n$. This measure is still denoted by dx, while the measure of a set $A \subseteq \mathbf{T}^n$ is denoted by $|A|$. Translation invariance of Lebesgue measure and the periodicity of functions on $\mathbf{T}^n$ imply that for all integrable functions f on $\mathbf{T}^n$, we have

$$\int_{\mathbf{T}^n} f(x)\,dx = \int_{[-1/2,1/2]^n} f(x)\,dx = \int_{[a_1,1+a_1]\times\cdots\times[a_n,1+a_n]} f(x)\,dx \tag{3.1.3}$$

for any real numbers $a_1,\dots,a_n$. In view of periodicity, integration by parts on the torus does not produce boundary terms; given f,g continuously differentiable functions on $\mathbf{T}^n$ we have

$$\int_{\mathbf{T}^n} \partial_j f(x)\,g(x)\,dx = -\int_{\mathbf{T}^n} \partial_j g(x)\,f(x)\,dx\,.$$

Elements of $\mathbf{Z}^n$ are denoted by $m=(m_1,\dots,m_n)$. For $m\in\mathbf{Z}^n$, we define the *total size* of m to be the number $|m| = (m_1^2+\cdots+m_n^2)^{1/2}$. Recall that for $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ in $\mathbf{R}^n$,

$$x\cdot y = x_1y_1+\cdots+x_ny_n$$

denotes the usual dot product. Finally, for $x\in\mathbf{T}^n$, $|x|$ denotes the usual Euclidean norm of x. If we identify $\mathbf{T}^n$ with $[-1/2,1/2]^n$, then $|x|$ can be interpreted as the distance of the element x from the origin, and then we have $0\le|x|\le\sqrt{n}/2$ for all $x\in\mathbf{T}^n$.

Multi-indices are elements of $(\mathbf{Z}^+\cup\{0\})^n$. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$, we denote the partial derivative $\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f$ by $\partial^\alpha f$. The spaces $\mathscr{C}^k(\mathbf{T}^n)$ of *continuously differentiable functions of order* k, where $k\in\mathbf{Z}^+$, are defined as the sets of functions φ for which $\partial^\alpha\varphi$ exist and are continuous for all $|\alpha|\le k$. When $k=0$ we set $\mathscr{C}^0(\mathbf{T}^n)=\mathscr{C}(\mathbf{T}^n)$ to be the space of continuous functions on $\mathbf{T}^n$. The space $\mathscr{C}^\infty(\mathbf{T}^n)$ of infinitely differentiable functions on $\mathbf{T}^n$ is the intersection of all the $\mathscr{C}^k(\mathbf{T}^n)$. All of these spaces are contained in $L^p(\mathbf{T}^n)$, which are nested, with $L^1(\mathbf{T}^n)$ being the largest.

3.1.2 Fourier Coefficients

Definition 3.1.1. For a complex-valued function f in $L^1(\mathbf{T}^n)$ and m in $\mathbf{Z}^n$, we define

$$\widehat{f}(m) = \int_{\mathbf{T}^n} f(x)e^{-2\pi i m\cdot x}dx\,. \tag{3.1.4}$$

We call $\widehat{f}(m)$ the mth *Fourier coefficient* of f. We note that $\widehat{f}(\xi)$ is not defined for $\xi\in\mathbf{R}^n\setminus\mathbf{Z}^n$, since the function $x\mapsto e^{-2\pi i\xi\cdot x}$ is not 1-periodic in any coordinate and therefore not well defined on $\mathbf{T}^n$. For a finite Borel measure μ on $\mathbf{T}^n$ and $m\in\mathbf{Z}^n$ the expression

$$\widehat{\mu}(m) = \int_{\mathbf{T}^n} e^{-2\pi i m\cdot x}d\mu \tag{3.1.5}$$

is called the mth *Fourier coefficient* of μ.