for some constant C, while the corresponding integral containing ψ_3 has arbitrary decay in r in view of estimate (2.6.3) (or Proposition 2.6.4 when $n=1$).

Exercises

2.6.1. Suppose that u is a real-valued $\mathscr{C}^k$ function defined on the line that satisfies $|u^{(k)}(t)| \ge c_0 > 0$ for some $k \ge 2$ and all $t \in (a,b)$. Prove that for $\lambda \in \mathbf{R}^+$ we have

$$\left| \int_a^b e^{i\lambda u(t)}\, dt \right| \le 12\, k\, (\lambda c_0)^{-1/k}$$

and that the same conclusion is valid when $k=1$, provided u' is monotonic.

2.6.2. Show that if u' is not monotonic in part (c) of Proposition 2.6.7, then the conclusion may fail.
[*Hint:* Let $\varphi(t)$ be a real-valued smooth function that is equal to $2t$ on intervals $[2\pi k+\varepsilon_k, 2\pi(k+\frac{1}{2})-\varepsilon_k]$ and equal to t on intervals $[2\pi(k+\frac{1}{2})+\varepsilon_k, 2\pi(k+1)-\varepsilon_k]$, where $0 \le k \le N$, for some $N \in \mathbf{Z}^+$. Show that the absolute value of the integral of $e^{i\varphi(t)}$ over the interval $[\varepsilon_0, 2\pi(N+1)-\varepsilon_N]$ tends to infinity as $N \to \infty$.]

2.6.3. Prove that the dependence on k of the constant in part (b) of Proposition 2.6.7 is indeed linear.
[*Hint:* Take $u(t) = t^k/k!$ over the interval $(0, k!)$.]

2.6.4. Follow the steps below to give an alternative proof of part (b) of Proposition 2.6.7. Assume that the statement is known for some $k \ge 2$ and some constant $C(k)$ for all intervals $[a,b]$ and all $\mathscr{C}^k$ functions satisfying $u^{(k)} \ge 1$ on $[a,b]$. Fix a $\mathscr{C}^{k+1}$ function u such that $u^{(k+1)} \ge 1$ on an interval $[a,b]$. Let c be the unique point at which the function $u^{(k+1)}$ attains its minimum in $[a,b]$.
(a) If $u^{(k)}(c) = 0$ for some $c \in (a,b)$, then for ~~all~~ $\delta > 0$ we have $u^{(k)}(t) \ge \delta$ in the complement of the interval $(c-\delta, c+\delta)$ and derive the bound

$$\left| \int_a^b e^{i\lambda u(t)} dt \right| \le 2C(k)(\lambda\delta)^{-1/k} + 2\delta\,.$$

(b) If $u^{(k)}(c) \ne 0$ for all $c \in (a,b)$, then we must have $c = a$. Obtain the bound

$$\left| \int_a^b e^{i\lambda u(t)} dt \right| \le C(k)(\lambda\delta)^{-1/k} + \delta\,.$$

(c) Choose a suitable δ to optimize and deduce the validity of the statement for $k+1$ with $C(k+1) = 2C(k)+2$, hence $C(k) = 3\cdot 2^{k-1} + 2^k - 2$, since $C(1) = 3$.

2.6.5. (a) Prove that for some constant C and all $\lambda \in \mathbf{R}$ and $\varepsilon \in (0,1)$ we have

$$\left| \int_{\varepsilon \le |t| \le 1} e^{i\lambda t} \frac{dt}{t} \right| \le C\,.$$

(b) Prove that for some $C' < \infty$, all $\lambda \in \mathbf{R}$, $k \in \mathbf{Z}^+$, and $\varepsilon \in (0,1)$ we have

$$\left| \int_{\varepsilon \le |t| \le 1} e^{i(\lambda t \pm t^k)} \frac{dt}{t} \right| \le C' .$$

(c) Show that there is a constant C'' such that for any $0 < \varepsilon < N < \infty$, for all ξ_1, ξ_2 in $\mathbf{R}$, and for all integers $k \ge 2$, we have

$$\left| \int_{\varepsilon \le |s| \le N} e^{i(\xi_1 s + \xi_2 s^k)} \frac{ds}{s} \right| \le C'' .$$

[*Hint:* Part (a): For $|\lambda|$ small use the inequality $|e^{i\lambda t} - 1| \le |\lambda t|$. If $|\lambda|$ is large, split the domains of integration into the regions $|t| \le |\lambda|^{-1}$ and $|t| \ge |\lambda|^{-1}$ and use integration by parts in the second case. Part (b): Write

$$\frac{e^{i(\lambda t \pm t^k)} - 1}{t} = e^{i\lambda t} \frac{e^{\pm i t^k} - 1}{t} + \frac{e^{i\lambda t}}{t}$$

and use part (a). Part (c): When $\xi_1 = \xi_2 = 0$ it is trivial. If $\xi_2 = 0$, $\xi_1 \ne 0$, change variables $t = \xi_1 s$ and then split the domain of integration into the sets $|t| \le 1$ and $|t| \ge 1$. In the interval over the set $|t| \le 1$ apply part (b) and over the set $|t| \ge 1$ use integration by parts. In the case $\xi_2 \ne 0$, change variables $t = |\xi_2|^{1/k} s$ and split the domain of integration into the sets $|t| \ge 1$ and $|t| \le 1$. When $|t| \le 1$ use part (b) and in the case $|t| \ge 1$ use Corollary 2.6.8, noting that $\frac{d^k(\xi_1 |\xi_2|^{-1/k} t \pm t^k)}{dt} = k! \ge 1$.]

2.6.6. (a) Show that for all $a > 0$ and $\lambda > 0$ the following is valid:

$$\left| \int_0^{a\lambda} e^{i\lambda \log t} \, dt \right| \le a .$$

(b) Prove that there is a constant $c > 0$ such that for all $b, \lambda > 1$ we have

$$\left| \int_0^{b} e^{i\lambda t \log t} \, dt \right| \le \frac{c}{\sqrt{\lambda}} .$$

[*Hint:* Part (b): Consider the intervals $(0, \delta)$ and $[\delta, b)$ for some δ. Apply Proposition 2.6.7 with $k = 1$ on one of these intervals and with $k = 2$ on the other. Then choose a suitable δ.]

2.6.7. Show that for all nonintegers $\gamma > 1$ there is a constant $C = C(\gamma) < \infty$ such that for all $\lambda, b > 0$ we have

$$\left| \int_0^{b} e^{i\lambda t^\gamma} dt \right| \le \frac{C}{\lambda^{1/\gamma}} .$$

[*Hint:* On the interval $(0, \delta)$ apply Proposition 2.6.7 with $k = [\gamma] + 1$ and on the interval (δ, b) with $k = [\gamma]$. Then optimize by choosing $\delta = \lambda^{-1/\gamma}$.]