where the equality $T(f)(x+y) = H(x+y)$ holds a.e. in y. Thus the continuous functions H_x and $\tau^{-x}H$ are equal a.e. and thus they must be everywhere equal, in particular, when $y=0$. This proves that $H_x(0) = H(x)$, which is a restatement of (2.5.3). $\square$

We now return to Lemmas 2.5.3 and 2.5.4. We begin with Lemma 2.5.3.

Proof. Consider first the multi-index $\alpha = (0,\dots,1,\dots,0)$, where 1 is in the jth entry and 0 is elsewhere. Let $e_j = (0,\dots,1,\dots,0)$, where 1 is in the jth entry and zero elsewhere. We have

$$\int_{\mathbf{R}^n} T(f)(y)\frac{\varphi(y+he_j)-\varphi(y)}{h}\,dy = \int_{\mathbf{R}^n} \varphi(y)T\left(\frac{\tau^{he_j}(f)-f}{h}\right)(y)\,dy \tag{2.5.4}$$

since both of these expressions are equal to

$$\int_{\mathbf{R}^n} \varphi(y)\frac{T(f)(y-he_j)-T(f)(y)}{h}\,dy$$

and T commutes with translations. We will let $h\to 0$ in both sides of (2.5.4). We write

$$\frac{\varphi(y+he_j)-\varphi(y)}{h} = \int_0^1 \partial_j\varphi(y+hte_j)\,dt\,,$$

from which it follows that for $|h|<1/2$ we have

$$\left|\frac{\varphi(y+he_j)-\varphi(y)}{h}\right| \le \int_0^1 \frac{C_M\,dt}{(1+|y+hte_j|)^M} \le \int_0^1 \frac{C_M\,dt}{(1+|y|-\frac{1}{2})^M} \le \frac{C_M'}{(|y|+1)^M}.$$

The integrand on the left-hand side of (2.5.4) is bounded by the integrable function $|T(f)(y)|C_M'(|y|+1)^{-M}$ and converges to $T(f)(y)\,\partial_j\varphi(y)$ as $h\to 0$. The Lebesgue dominated convergence theorem yields that the integral on the left-hand side of (2.5.4) converges to

$$\int_{\mathbf{R}^n} T(f)(y)\,\partial_j\varphi(y)\,dy = -\langle \partial_j T(f),\varphi\rangle, \tag{2.5.5}$$

where $\partial_j T(f)$ is the distributional derivative of $T(f)$. Moreover, for a Schwartz function f we have

$$\frac{\tau^{he_j}(f)(y)-f(y)}{-h} = \int_0^1 \partial_j f(y-hte_j)\,dt\,,$$

which converges to $\partial_j f(y)$ pointwise as $h\to 0$ and is bounded by $C_M'(1+|y|)^{-M}$ for $|h|<1/2$ by an argument similar to the preceding one for φ in place of f. Thus

$$\frac{\tau^{he_j}(f)-f}{-h} \to \partial_j f \quad \text{in } L^p \text{ as } h\to 0, \tag{2.5.6}$$

by the Lebesgue dominated convergence theorem. The boundedness of T from L^p to L^q yields that

$$T\left(\frac{\tau^{he_j}(f)-f}{h}\right) \to -T(\partial_j f) \quad \text{in } L^q \text{ as } h\to 0. \tag{2.5.7}$$

Since $\varphi \in L^{q'}$, by Hölder's inequality, the right-hand side of (2.5.4) converges to

$$-\int_{\mathbf{R}^n} \varphi(y) T(\partial_j f)(y)\,dy = -\left\langle T(\partial_j f), \varphi\right\rangle$$

as $h\to 0$. This limit is equal to (2.5.5) and the required conclusion follows for $\alpha = (0,\dots,0,1,0,\dots,0)$. The general case follows by induction on $|\alpha|$. $\square$

We now prove Lemma 2.5.4.

Proof. Let $R\geq 1$. Fix a $\mathscr{C}_0^\infty$ function φ_R that is equal to 1 in the ball $|x|\leq R$ and equal to zero when $|x|\geq 2R$. Since h is in $L^q(\mathbf{R}^n)$, it follows that $\varphi_R h$ is in $L^1(\mathbf{R}^n)$. We show that $\widehat{\varphi_R h}$ is also in L^1. We begin with the inequality

$$1 \leq C_n(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} |(-2\pi i x)^\alpha|, \tag{2.5.8}$$

which is just a restatement of (2.2.3). Now multiply (2.5.8) by $|\widehat{\varphi_R h}(x)|$ to obtain

$$\begin{aligned}
|\widehat{\varphi_R h}(x)| &\leq C_n(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} |(-2\pi i x)^\alpha \widehat{\varphi_R h}(x)| \\
&\leq C_n(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} \big\|(\partial^\alpha(\varphi_R h))^\wedge\big\|_{L^\infty} \\
&\leq C_n(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} \big\|\partial^\alpha(\varphi_R h)\big\|_{L^1} \\
&\leq C_n(2^n R^n v_n)^{1/q'}(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} \big\|\partial^\alpha(\varphi_R h)\big\|_{L^q} \\
&\leq C_{n,R}(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} \big\|\partial^\alpha h\big\|_{L^q},
\end{aligned}$$

where we used Leibniz's rule (Proposition 2.3.22 (14)) and the fact that all derivatives of φ_R are pointwise bounded by constants depending on R.

Integrate the previously displayed inequality with respect to x to obtain

$$\big\|\widehat{\varphi_R h}\big\|_{L^1} \leq C_{R,n} \sum_{|\alpha|\leq n+1} \big\|\partial^\alpha h\big\|_{L^q} < \infty. \tag{2.5.9}$$

Therefore, Fourier inversion holds for $\varphi_R h$ (see Exercise 2.2.6). This implies that $\varphi_R h$ is equal a.e. to a continuous function, namely the inverse Fourier transform of its Fourier transform. Since $\varphi_R = 1$ on the ball $B(0,R)$, we conclude that h is a.e. equal to a continuous function in this ball. Since $R>0$ was arbitrary, it follows that