Exercises

2.5.1. Prove that if $f \in L^q(\mathbf{R}^n)$ and $0 < q < \infty$, then

$$\left\|\tau^h(f)+f\right\|_{L^q} \to 2^{1/q}\left\|f\right\|_{L^q} \qquad \text{as } |h| \to \infty.$$

2.5.2. Prove Proposition 2.5.14. Also prove that if $\delta_j^{h_j}$ is a dilation operator in the jth variable (for instance $\delta_1^{h_1} f(x) = f(h_1x_1, x_2, \dots, x_n)$), then

$$\left\|\delta_1^{h_1}\cdots\delta_n^{h_n} m\right\|_{\mathscr{M}_p} = \left\|m\right\|_{\mathscr{M}_p}.$$

2.5.3. Let $m \in \mathscr{M}_p(\mathbf{R}^n)$ where $1 \le p < \infty$.
(a) If ψ is a function on $\mathbf{R}^n$ whose inverse Fourier transform is an integrable function, then prove that

$$\left\|\psi m\right\|_{\mathscr{M}_p} \le \left\|\psi^\vee\right\|_{L^1}\left\|m\right\|_{\mathscr{M}_p}.$$

(b) If ψ is in $L^1(\mathbf{R}^n)$, then prove that

$$\left\|\psi * m\right\|_{\mathscr{M}_p} \le \left\|\psi\right\|_{L^1}\left\|m\right\|_{\mathscr{M}_p}.$$

2.5.4. Fix a multi-index γ and $1 \le p,q < \infty$.
(a) Prove that the map $T(f) = f * \partial^\gamma \delta_0$ maps $\mathscr{S}$ continuously into $\mathscr{S}$.
(b) Prove that when $|\gamma| > 0$, then T does not extend to an element of the space $\mathscr{M}^{p,q}$.

2.5.5. Let $K_\gamma(x) = |x|^{-n+\gamma}$, where $0 < \gamma < n$. Use Theorem 1.4.25 to show that the operator

$$T_\gamma(f) = f * K_\gamma, \qquad f \in \mathscr{S},$$

extends to a bounded operator in $\mathscr{M}^{p,q}(\mathbf{R}^n)$, where $1/p - 1/q = \gamma/n$, $1 < p < q < \infty$. This provides an example of a nontrivial operator in $\mathscr{M}^{p,q}(\mathbf{R}^n)$ when $p < q$.

2.5.6. (a) Use the ideas of the proof of Proposition 2.5.13 to show that if $m_j \in \mathscr{M}_p$, $1 \le p < \infty$, $\|m_j\|_{\mathscr{M}_p} \le C$ for all $j = 1, 2, \dots$, and $m_j \to m$ a.e., then $m \in \mathscr{M}_p$ and

$$\left\|m\right\|_{\mathscr{M}_p(\mathbf{R}^n)} \le \liminf_{j\to\infty}\left\|m_j\right\|_{\mathscr{M}_p(\mathbf{R}^n)} \le C.$$

(b) Prove that if $m \in \mathscr{M}_p$, $1 \le p < \infty$, and the limit $m_0(\xi) = \lim_{R\to\infty} m(\xi/R)$ exists for all $\xi \in \mathbf{R}^n$, then m_0 is a radial function in $\mathscr{M}_p(\mathbf{R}^n)$ and satisfies $\|m_0\|_{\mathscr{M}_p} \le \|m\|_{\mathscr{M}_p}$.
(c) If $m \in \mathscr{M}_p(\mathbf{R})$ has left and right limits at the origin, then prove that

$$\left\|m\right\|_{\mathscr{M}_p(\mathbf{R})} \ge \max(|m(0+)|, |m(0-)|).$$