It is a consequence of Theorem 1.3.4 that the normed spaces $\mathscr{M}_p$ are nested, that is, for $1 \le p \le q \le 2$ we have

$$\mathscr{M}_1 \subseteqq \mathscr{M}_p \subseteqq \mathscr{M}_q \subseteqq \mathscr{M}_2 = L^\infty .$$

Moreover, if $m \in \mathscr{M}_p$ and $1 \le p \le 2 \le p'$, Theorem 1.3.4 gives

$$\big\|T_m\big\|_{L^2\to L^2} \le \big\|T_m\big\|_{L^p\to L^p}^{\frac{1}{2}} \big\|T_m\big\|_{L^{p'}\to L^{p'}}^{\frac{1}{2}} = \big\|T_m\big\|_{L^p\to L^p} , \tag{2.5.16}$$

since $1/2 = (1/2)/p + (1/2)/p'$. Theorem 1.3.4 also gives that

$$\big\|m\big\|_{\mathscr{M}_q} \le \big\|m\big\|_{\mathscr{M}_p}$$

whenever $1 \le q \le p \le 2$. Thus the $\mathscr{M}_p$'s form an increasing family of spaces as p increases from 1 to 2.

Example 2.5.12. The function $m(\xi) = e^{2\pi i \xi \cdot b}$ is an L^p multiplier for all $b \in \mathbf{R}^n$, since the corresponding operator $T_m(f)(x) = f(x+b)$ is bounded on $L^p(\mathbf{R}^n)$. Clearly $\|m\|_{\mathscr{M}_p} = 1$.

Proposition 2.5.13. *For $1 \le p < \infty$, the normed space $\big(\mathscr{M}_p, \|\cdot\|_{\mathscr{M}_p}\big)$ is a Banach space. Furthermore, $\mathscr{M}_p$ is closed under pointwise multiplication and is a Banach algebra.*

Proof. It suffices to consider the case $1 \le p \le 2$. It is straightforward that if m_1, m_2 are in $\mathscr{M}_p$ and $b \in \mathbf{C}$ then $m_1 + m_2$ and bm_1 are also in $\mathscr{M}_p$. Observe that $m_1 m_2$ is the multiplier that corresponds to the operator $T_{m_1} T_{m_2} = T_{m_1 m_2}$ and thus

$$\big\|m_1 m_2\big\|_{\mathscr{M}_p} = \big\|T_{m_1} T_{m_2}\big\|_{L^p\to L^p} \le \big\|m_1\big\|_{\mathscr{M}_p} \big\|m_2\big\|_{\mathscr{M}_p} .$$

This proves that $\mathscr{M}_p$ is an algebra. To show that $\mathscr{M}_p$ is a complete normed space, consider a Cauchy sequence m_j in $\mathscr{M}_p$. It follows from (2.5.16) that m_j is Cauchy in L^∞, and hence it converges to some bounded function m in the L^∞ norm; moreover all the m_j are a.e. bounded by some constant C uniformly in j. We have to show that $m \in \mathscr{M}_p$. Fix $f \in \mathscr{S}$. We have

$$T_{m_j}(f)(x) = \int_{\mathbf{R}^n} \widehat{f}(\xi) m_j(\xi) e^{2\pi i x\cdot\xi}\, d\xi \to \int_{\mathbf{R}^n} \widehat{f}(\xi) m(\xi) e^{2\pi i x\cdot\xi}\, d\xi = T_m(f)(x)$$

a.e. by the Lebesgue dominated convergence theorem, since $C|\widehat{f}\,|$ is an integrable upper bound of all integrands on the left in the preceding expression. Since $\{m_j\}_j$ is a Cauchy sequence in $\mathscr{M}_p$, it is bounded in $\mathscr{M}_p$, and thus $\sup_j \|m_j\|_{\mathscr{M}_p} < +\infty$. An application of Fatou's lemma yields that

$$\begin{aligned}
\int_{\mathbf{R}^n} |T_m(f)|^p\, dx &= \int_{\mathbf{R}^n} \liminf_{j\to\infty} |T_{m_j}(f)|^p\, dx \\
&\le \liminf_{j\to\infty} \int_{\mathbf{R}^n} |T_{m_j}(f)|^p\, dx \\
&\le \liminf_{j\to\infty} \big\|m_j\big\|_{\mathscr{M}_p}^p \big\|f\big\|_{L^p}^p ,
\end{aligned}$$