2.5.4 Characterizations of $\mathscr{M}^{1,1}(\mathbf{R}^n)$ and $\mathscr{M}^{2,2}(\mathbf{R}^n)$

It would be desirable to have a characterization of the spaces $\mathscr{M}^{p,p}$ in terms of properties of the convolving distribution. Unfortunately, this is unknown at present (it is not clear whether it is possible) except for certain cases.

Theorem 2.5.8. *An operator T is in $\mathscr{M}^{1,1}(\mathbf{R}^n)$ if and only if it is given by convolution with a finite Borel (complex-valued) measure. In this case, the norm of the operator is equal to the total variation of the measure.*

Proof. If T is given by convolution with a finite Borel measure μ, then clearly T maps L^1 to itself and $\|T\|_{L^1\to L^1} \le \|\mu\|_{\mathscr{M}}$, where $\|\mu\|_{\mathscr{M}}$ is the total variation of μ.

Conversely, let T be an operator bounded from L^1 to L^1 that commutes with translations. By Theorem 2.5.2, T is given by convolution with a tempered distribution u. Let

$$f_\varepsilon(x) = \varepsilon^{-n} e^{-\pi|x/\varepsilon|^2}.$$

Since the functions f_ε are uniformly bounded in L^1, it follows from the boundedness of T that $f_\varepsilon * u$ are also uniformly bounded in L^1. Since L^1 is naturally embedded in the space of finite Borel measures, which is the dual of the space $\mathscr{C}_{00}$ of continuous functions that tend to zero at infinity, we obtain that the family $f_\varepsilon * u$ lies in a fixed multiple of the unit ball of $\mathscr{C}_{00}^*$. By the Banach–Alaoglu theorem, this is a weak* compact set. Therefore, some subsequence of $f_\varepsilon * u$ converges in the weak* topology to a measure μ. That is, for some $\varepsilon_k \to 0$ and all $g \in \mathscr{C}_{00}(\mathbf{R}^n)$ we have

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} g(x)(f_{\varepsilon_k} * u)(x)\,dx = \int_{\mathbf{R}^n} g(x)\,d\mu(x). \tag{2.5.13}$$

We claim that $u = \mu$. To see this, fix $g \in \mathscr{S}$. Equation (2.5.13) implies that

$$\big\langle u, \widetilde{f_{\varepsilon_k}} * g\big\rangle = \big\langle u, f_{\varepsilon_k} * g\big\rangle \to \big\langle \mu, g\big\rangle$$

as $k \to \infty$. Exercise 2.3.2 gives that $g * f_{\varepsilon_k}$ converges to g in $\mathscr{S}$. Therefore,

$$\big\langle u, f_{\varepsilon_k} * g\big\rangle \to \big\langle u, g\big\rangle.$$

It follows from (2.5.13) that $\langle u, g\rangle = \langle \mu, g\rangle$, and since g was arbitrary, $u = \mu$.

Next, (2.5.13) implies that for all $g \in \mathscr{C}_{00}$ we have

$$\left| \int_{\mathbf{R}^n} g(x)\,d\mu(x) \right| \le \big\|g\big\|_{L^\infty} \sup_k \big\|f_{\varepsilon_k} * u\big\|_{L^1} \le \big\|g\big\|_{L^\infty} \big\|T\big\|_{L^1\to L^1}. \tag{2.5.14}$$

The Riesz representation theorem gives that the norm of the functional

$$g \mapsto \int_{\mathbf{R}^n} g(x)\,d\mu(x)$$

on $\mathscr{C}_{00}$ is exactly $\|\mu\|_{\mathscr{M}}$. It follows from (2.5.14) that $\|T\|_{L^1\to L^1} \ge \|\mu\|_{\mathscr{M}}$. Since the reverse inequality is obvious, we conclude that $\|T\|_{L^1\to L^1} = \|\mu\|_{\mathscr{M}}$. □