h is a.e. equal to a continuous function on $\mathbf{R}^n$, which we denote by H. Finally, (2.5.2) is a direct consequence of (2.5.9) with $R=1$, since $|H(0)| \le \|\widehat{\varphi_1 h}\|_{L^1}$. $\square$

2.5.2 The Transpose and the Adjoint of a Linear Operator

We briefly discuss the notions of the transpose and the adjoint of a linear operator. We first recall real and complex inner products. For f,g measurable functions on $\mathbf{R}^n$, we define the *complex inner product*

$$\langle f \,|\, g\rangle = \int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx\,,$$

whenever the integral converges absolutely. We reserve the notation

$$\langle f, g\rangle = \int_{\mathbf{R}^n} f(x)g(x)\,dx$$

for the *real inner product* on $L^2(\mathbf{R}^n)$ and also for the action of a distribution f on a test function g. (This notation also makes sense when a distribution f coincides with a function.)

Let $1 \le p,q \le \infty$. For a bounded linear operator T from $L^p(X,\mu)$ to $L^q(Y,\nu)$ we denote by T^* its *adjoint operator* defined by

$$\langle T(f) \,|\, g\rangle = \int_Y T(f)\,\overline{g}\,d\nu = \int_X f\,\overline{T^*(g)}\,d\mu = \langle f \,|\, T^*(g)\rangle \tag{2.5.10}$$

for f in $L^p(X,\mu)$ and g in $L^{q'}(Y,\nu)$ (or in a dense subspace of it). We also define the *transpose* of T as the unique operator T^t that satisfies

$$\langle T(f), g\rangle = \int_Y T(f)\,g\,d\nu = \int_X f\,T^t(g)\,d\mu = \langle f, T^t(g)\rangle$$

for all $f \in L^p(X,\mu)$ and all $g \in L^{q'}(Y,\nu)$.

If T is an integral operator of the form

$$T(f)(x) = \int_X K(x,y)f(y)\,d\mu(y),$$

then T^* and T^t are also integral operators with kernels $K^*(x,y) = \overline{K(y,x)}$ and $K^t(x,y) = K(y,x)$, respectively. If T has the form $T(f) = (\widehat{f}m)^\vee$, that is, it is given by multiplication on the Fourier transform by a (complex-valued) function $m(\xi)$, then T^* is given by multiplication on the Fourier transform by the function $\overline{m(\xi)}$. Indeed for f,g in $\mathscr{S}(\mathbf{R}^n)$ we have