$\big[$*Hint:* Part (a): See Appendix D.4 Part (b): Use the stereographic projection in Appendix D.6.$\big]$

2.4.9. Prove the following *beta integral identity:*

$$\int_{\mathbf{R}^n} \frac{dt}{|x-t|^{\alpha_1}|y-t|^{\alpha_2}} = \pi^{\frac{n}{2}} \frac{\Gamma\big(\frac{n-\alpha_1}{2}\big)\Gamma\big(\frac{n-\alpha_2}{2}\big)\Gamma\big(\frac{\alpha_1+\alpha_2-n}{2}\big)}{\Gamma\big(\frac{\alpha_1}{2}\big)\Gamma\big(\frac{\alpha_2}{2}\big)\Gamma\big(n-\frac{\alpha_1+\alpha_2}{2}\big)} |x-y|^{n-\alpha_1-\alpha_2},$$

where $0<\alpha_1,\alpha_2<n$, $\alpha_1+\alpha_2>n$.
$\big[$*Hint:* Reduce to the case $y=0$, interpret the integral as a convolution, and use Theorem 2.4.6.$\big]$

2.4.10. (a) Prove that if an ~~a continuous~~ integrable function f on $\mathbf{R}^n$ $(n\ge 2)$ is constant on the spheres $r\mathbf{S}^{n-1}$ for all $r>0$, then so is its Fourier transform.
(b) If a continuous integrable function on $\mathbf{R}^n$ $(n\ge 3)$ is constant on all $(n-2)$–dimensional spheres orthogonal to $e_1=(1,0,\dots,0)$, then its Fourier transform has the same property.

2.4.11. ([137]) Suppose that $0<d_1,d_2,d_3<n$ satisfy $d_1+d_2+d_3=2n$. Prove that for any distinct $x,y,z\in\mathbf{R}^n$ we have the identity

$$\begin{aligned}&\int_{\mathbf{R}^n} |x-t|^{-d_2}|y-t|^{-d_3}|z-t|^{-d_1}\,dt\\ &\qquad = \pi^{\frac{n}{2}}\left(\prod_{j=1}^{3}\frac{\Gamma\big(n-\frac{d_j}{2}\big)}{\Gamma\big(\frac{d_j}{2}\big)}\right)|x-y|^{d_1-n}|y-z|^{d_2-n}|z-x|^{d_3-n}.\end{aligned}$$

$\big[$*Hint:* Reduce matters to the case that $z=0$ and $y=e_1$. Then take the Fourier transform in x and use that the function $h(t)=|t-e_1|^{-d_3}|t|^{-d_1}$ satisfies $\widehat{h}(\xi)=\widehat{h}(A_\xi^{-2}\xi)$ for all $\xi\neq 0$, where A_ξ is an orthogonal matrix with $A_\xi e_1=\xi/|\xi|$.$\big]$

2.4.12. (a) Integrate the function e^{iz^2} over the contour consisting of the three pieces $P_1=\{x+i0:\ 0\le x\le R\}$, $P_2=\{Re^{i\theta}:\ 0\le\theta\le\frac{\pi}{4}\}$, and $P_3=\{r\,e^{i\frac{\pi}{4}}:\ 0\le r\le R\}$ (with the proper orientation) to obtain the *Fresnel integral identity*:

$$\lim_{R\to\infty}\int_0^R e^{ix^2}\,dx=\tfrac{\sqrt{2\pi}}{4}(1+i)\,.$$

(b) Use the result in part (a) to show that the Fourier transform of the function $e^{i\pi|x|^2}$ in $\mathbf{R}^n$ is equal to $e^{i\frac{\pi n}{4}}e^{-i\pi|\xi|^2}$.
$\big[$*Hint:* Part (a): On P_2 we have $e^{-R^2\sin(2\theta)}\le e^{-\frac{4}{\pi}R^2\theta}$, and the integral over P_2 tends to 0. Part (b): Try first $n=1$.$\big]$