To prove this result we need the following proposition, whose proof we postpone until the end of this section.

Proposition 2.4.8. *Suppose that u in $\mathscr{S}'(\mathbf{R}^n)$ is a $\mathscr{C}^\infty$ function on $\mathbf{R}^n \setminus \{0\}$ and homogeneous of degree $z \in \mathbf{C}$. Then $\widehat{u}$ is a $\mathscr{C}^\infty$ function on $\mathbf{R}^n \setminus \{0\}$.*

We now prove Proposition 2.4.7 using Proposition 2.4.8.

Proof. Let a be the integral of the smooth function m over $\mathbf{S}^{n-1}$. The function $m-a$ is homogeneous of degree zero and thus locally integrable on $\mathbf{R}^n$; hence it can be thought of as a tempered distribution that we call $\widehat{u}$ (the Fourier transform of a tempered distribution u). Since $\widehat{u}$ is a $\mathscr{C}^\infty$ function on $\mathbf{R}^n \setminus \{0\}$, Proposition 2.4.8 implies that u is also a $\mathscr{C}^\infty$ function on $\mathbf{R}^n \setminus \{0\}$. Let Ω be the restriction of u on $\mathbf{S}^{n-1}$. Then Ω is a well defined $\mathscr{C}^\infty$ function on $\mathbf{S}^{n-1}$. Since u is a homogeneous function of degree $-n$ that coincides with the smooth function Ω on $\mathbf{S}^{n-1}$, it follows that $u(x) = \Omega(x/|x|)/|x|^n$ for x in $\mathbf{R}^n \setminus \{0\}$.

We show that Ω has mean value zero over $\mathbf{S}^{n-1}$. Pick a nonnegative, radial, smooth, and nonzero function ψ on $\mathbf{R}^n$ supported in the annulus $1 < |x| < 2$. Switching to polar coordinates, we write

$$\begin{aligned}\langle u, \psi \rangle &= \int_{\mathbf{R}^n} \frac{\Omega(x/|x|)}{|x|^n} \psi(x)\, dx = c_\psi \int_{\mathbf{S}^{n-1}} \Omega(\theta)\, d\theta, \\ \langle u, \psi \rangle &= \langle \widehat{u}, \widehat{\psi} \rangle = \int_{\mathbf{R}^n} (m(\xi) - a) \widehat{\psi}(\xi)\, d\xi = c'_\psi \int_{\mathbf{S}^{n-1}} \big(m(\theta) - a\big)\, d\theta = 0,\end{aligned}$$

and thus Ω has mean value zero over $\mathbf{S}^{n-1}$ (since $c_\psi \neq 0$).

We can now legitimately define the distribution W_Ω, which coincides with the function $\Omega(x/|x|)/|x|^n$ on $\mathbf{R}^n \setminus \{0\}$. But the distribution u also coincides with this function on $\mathbf{R}^n \setminus \{0\}$. It follows that $u - W_\Omega$ is supported at the origin. Proposition 2.4.1 now gives that $u - W_\Omega$ is a sum of derivatives of Dirac masses. Since both distributions are homogeneous of degree $-n$, it follows that

$$u - W_\Omega = c\delta_0 .$$

But $u = (m-a)^\vee = m^\vee - a\delta_0$, and thus $m^\vee = (c+a)\delta_0 + W_\Omega$. This proves the proposition. $\square$

We now turn to the proof of Proposition 2.4.8.

Proof. Let $u \in \mathscr{S}'$ be homogeneous of degree z and $\mathscr{C}^\infty$ on $\mathbf{R}^n \setminus \{0\}$. We need to show that $\widehat{u}$ is $\mathscr{C}^\infty$ away from the origin. We prove that $\widehat{u}$ is $\mathscr{C}^M$ for all M. Fix $M \in \mathbf{Z}^+$ and let α be any multi-index such that

$$|\alpha| > n + M + \operatorname{Re} z . \tag{2.4.14}$$