$\big[$*Hint:* Let f_n be a Cauchy sequence in L^p. Pass to a subsequence $\{n_i\}_i$ such that $\|f_{n_{i+1}} - f_{n_i}\|_{L^p} \le 2^{-i}$. Then the series $f = f_{n_1} + \sum_{i=1}^{\infty}(f_{n_{i+1}} - f_{n_i})$ converges in L^p.$\big]$

1.1.9. Let (X,μ) be a measure space with $\mu(X) < \infty$. Suppose that a sequence of measurable functions f_n on X converges to f μ-a.e. Prove that f_n converges to f in measure.
$\big[$*Hint:* For $\varepsilon > 0$, $\big\{x \in X:\ f_n(x) \to f(x)\big\} \subseteq \bigcup\limits_{m=1}^{\infty} \bigcap\limits_{n=m}^{\infty} \big\{x \in X:\ |f_n(x) - f(x)| < \varepsilon\big\}$.$\big]$

1.1.10. Let f be a measurable function on (X,μ) such that $d_f(\alpha) < \infty$ for all $\alpha > 0$. Fix $\gamma > 0$ and define $f_\gamma = f\chi_{|f|>\gamma}$ and $f^\gamma = f - f_\gamma = f\chi_{|f|\le\gamma}$.
(a) Prove that

$$d_{f_\gamma}(\alpha) = \begin{cases} d_f(\alpha) & \text{when} \quad \alpha > \gamma, \\ d_f(\gamma) & \text{when} \quad \alpha \le \gamma, \end{cases}$$
$$d_{f^\gamma}(\alpha) = \begin{cases} 0 & \text{when} \quad \alpha \ge \gamma, \\ d_f(\alpha) - d_f(\gamma) & \text{when} \quad \alpha < \gamma. \end{cases}$$

(b) If $f \in L^p(X,\mu)$ then

$$\begin{aligned} \big\|f_\gamma\big\|_{L^p}^p &= p\int_\gamma^\infty \alpha^{p-1} d_f(\alpha)\,d\alpha + \gamma^p d_f(\gamma), \\ \big\|f^\gamma\big\|_{L^p}^p &= p\int_0^\gamma \alpha^{p-1} d_f(\alpha)\,d\alpha - \gamma^p d_f(\gamma), \\ \int_{\gamma<|f|\le\delta} |f|^p\,d\mu &= p\int_\gamma^\delta d_f(\alpha)\alpha^{p-1}\,d\alpha - \delta^p d_f(\delta) + \gamma^p d_f(\gamma). \end{aligned}$$

(c) If f is in $L^{p,\infty}(X,\mu)$ prove that f^γ is in $L^q(X,\mu)$ for any $q > p$ and f_γ is in $L^q(X,\mu)$ for any $q < p$. Thus $L^{p,\infty} \subseteq L^{p_0} + L^{p_1}$ when $0 < p_0 < p < p_1 \le \infty$.

1.1.11. Let (X,μ) be a measure space and let E be a subset of X with $\mu(E) < \infty$. Assume that f is in $L^{p,\infty}(X,\mu)$ for some $0 < p < \infty$.
(a) Show that for $0 < q < p$ we have

$$\int_E |f(x)|^q\,d\mu(x) \le \frac{p}{p-q}\mu(E)^{1-\frac{q}{p}}\big\|f\big\|_{L^{p,\infty}}^q.$$

(b) Conclude that if $\mu(X) < \infty$ and $0 < q < p$, then

$$L^p(X,\mu) \subseteq L^{p,\infty}(X,\mu) \subseteq L^q(X,\mu).$$

$\big[$*Hint:* Part (a): Use $\mu\big(E \cap \{|f| > \alpha\}\big) \le \min\big(\mu(E), \alpha^{-p}\big\|f\big\|_{L^{p,\infty}}^p\big)$.$\big]$

1.1.12. (*Normability of weak L^p for $p > 1$*) Let (X,μ) be a σ-finite measure space and let $0 < p < \infty$. Pick $0 < r < p$ and define

$$|||\, f\, |||_{L^{p,\infty}} = \sup_{0<\mu(E)<\infty} \mu(E)^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E |f|^r d\mu\right)^{\frac{1}{r}},$$

where the supremum is taken over all measurable subsets E of X of finite measure.
(a) Use Exercise 1.1.11 with $q = r$ to conclude that

$$|||\, f \,|||_{L^{p,\infty}} \le \left(\frac{p}{p-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}}$$

for all f in $L^{p,\infty}(X,\mu)$. (It is not needed that X be σ-finite here).
(b) Prove that for every measurable function f on (X,μ) we have

$$\|f\|_{L^{p,\infty}} \le |||\, f \,|||_{L^{p,\infty}}.$$

(Y. Oi) Notice that if $X = \{1,2\}$, $\mu(\{1\}) = 1$, $\mu(\{2\}) = \infty$, then X is not σ-finite, and verify that for the function $f = 1$ the preceding inequality fails.
(c) Show that $L^{p,\infty}(X,\mu)$ is metrizable for all $0 < p < \infty$, i.e., there is a metric on the space that generates the same topology as the quasi-norm. Also show that $L^{p,\infty}(X,\mu)$ is *normable* when $p > 1$, i.e., there is a norm on the space equivalent to $\|\cdot\|_{L^{p,\infty}}$.
(d) Use the characterization of the weak L^p quasi-norm obtained in parts (a) and (b) to prove Fatou's lemma for this space: For all measurable functions g_n on X we have

$$\Big\|\liminf_{n\to\infty}|g_n|\Big\|_{L^{p,\infty}} \le C_p \liminf_{n\to\infty}\big\|g_n\big\|_{L^{p,\infty}}$$

for some constant C_p that depends only on $p \in (0,\infty)$.
$\big[$*Hint:* Part (b): Write $X = \bigcup_{k=1}^{\infty} X_k$ with $\mu(X_k) < \infty$ and take $E = \{|f| > \alpha\} \cap X_k$.$\big]$

1.1.13. Consider the $N!$ functions on the line

$$f_\sigma = \sum_{j=1}^{N} \frac{N}{\sigma(j)} \chi_{[\frac{j-1}{N},\frac{j}{N})},$$

where σ is a permutation of the set $\{1,2,\dots,N\}$.
(a) Show that each f_σ satisfies $\|f_\sigma\|_{L^{1,\infty}} = 1$.
(b) Show that $\|\sum_{\sigma\in S_N} f_\sigma\|_{L^{1,\infty}} = N!\big(1+\frac{1}{2}+\cdots+\frac{1}{N}\big)$.
(c) Conclude that the space $L^{1,\infty}(\mathbf{R})$ is not normable (this means that $\|\cdot\|_{L^{1,\infty}}$ is not equivalent to a norm).
(d) Use a similar argument to prove that $L^{1,\infty}(\mathbf{R}^n)$ is not normable by considering the functions

$$f_\sigma(x_1,\dots,x_n) = \sum_{j_1=1}^{N}\cdots\sum_{j_n=1}^{N} \frac{N^n}{\sigma(\tau(j_1,\dots,j_n))} \chi_{[\frac{j_1-1}{N},\frac{j_1}{N})}(x_1)\cdots\chi_{[\frac{j_n-1}{N},\frac{j_n}{N})}(x_n),$$

where σ is a permutation of the set $\{1,2,\dots,N^n\}$ and τ is a fixed injective map from the set of all n-tuples of integers with coordinates $1 \le j \le N$ onto the set $\{1,2,\dots,N^n\}$. One may take

$$\tau(j_1,\dots,j_n) = j_1 + N(j_2-1) + N^2(j_3-1) + \cdots + N^{n-1}(j_n-1),$$

for instance.