To justify the fact concerning the convergence of the Riemann sums, we argue as in the proof of the previous theorem. For each $N = 1,2,\dots$, consider a partition of $[-N,N]^n$ into $(2N^2)^n$ cubes Q_m of side length $1/N$ and let y_m be the center of each Q_m. For a multi-index α let

$$D_N(\xi) = \sum_{m=1}^{(2N^2)^n} f(y_m)(-2\pi i y_m)^\alpha e^{-2\pi i y_m\cdot\xi}|Q_m| - \int_{\mathbf{R}^n} f(x)(-2\pi i x)^\alpha e^{-2\pi i x\cdot\xi}\,dx\,.$$

We must show that for every $M>0$, $\sup_{|\xi|\le M}|D_N(\xi)|$ converges to zero as $N\to\infty$. Setting $g(x) = f(x)(-2\pi i x)^\alpha$, we write

$$D_N(\xi) = \sum_{m=1}^{(2N^2)^n} \int_{Q_m} \big[g(y_m)e^{-2\pi i y_m\cdot\xi} - g(x)e^{-2\pi i x\cdot\xi}\big]\,dx - \int_{([-N,N]^n)^c} g(x)e^{-2\pi i x\cdot\xi}\,dx\,.$$

Using the mean value theorem, we bound the absolute value of the expression inside the square brackets by

$$\big(|\nabla g(z_m)| + 2\pi|\xi|\,|g(z_m)|\big)\frac{\sqrt{n}}{N} \le \frac{C_K\,(1+|\xi|)}{(1+|z_m|)^K}\,\frac{\sqrt{n}}{N}\,,$$

for some point z_m in the cube Q_m. Since

$$\sum_{m=1}^{(2N^2)^n} \int_{Q_m} \frac{C_K\,(1+|\xi|)}{(1+|z_m|)^K}\,dx \le C_K'(1+M) < \infty$$

for $|\xi|\le M$, it follows that $\sup_{|\xi|\le M}|D_N(\xi)|\to 0$ as $N\to\infty$. □

Next we give a proposition that extends the properties of the Fourier transform to tempered distributions.

Proposition 2.3.22. *Given u, v in $\mathscr{S}'(\mathbf{R}^n)$, $f_j, f\in\mathscr{S}$, $y\in\mathbf{R}^n$, b a complex scalar, α a multi-index, and $a>0$, we have*

(1) $\widehat{u+v} = \widehat{u}+\widehat{v}$,

(2) $\widehat{bu} = b\widehat{u}$,

(3) *If $f_j\to f$ in $\mathscr{S}$, then $\widehat{f_j}\to\widehat{f}$ in $\mathscr{S}$ and if $u_j\to u$ in $\mathscr{S}'$, then $\widehat{u}_j\to\widehat{u}$ in $\mathscr{S}'$,*

(4) $(\widetilde{u})\widehat{\ } = (\widehat{u})\widetilde{\ }$,

(5) $(\tau^y u)\widehat{\ } = e^{-2\pi i y\cdot\xi}\widehat{u}$,

(6) $(e^{2\pi i x\cdot y}u)\widehat{\ } = \tau^y\widehat{u}$,

(7) $(\delta^a u)\widehat{\ } = (\widehat{u})_a = a^{-n}\delta^{a^{-1}}\widehat{u}$,

(8) $(\partial^\alpha u)\widehat{\ } = (2\pi i\xi)^\alpha\widehat{u}$,

(9) $\partial^\alpha\widehat{u} = ((-2\pi i x)^\alpha u)\widehat{\ }$,