But the second integral in the preceding expression is bounded by

$$\int_{([-N,N]^n)^c} \frac{C'''|x|^{|\alpha|}}{(1+|x-y|)^{M/2}} \frac{dy}{(1+|y|)^M} \le \frac{C'''|x|^{|\alpha|}}{(1+|x|)^{M/2}} \int_{([-N,N]^n)^c} \frac{dy}{(1+|y|)^{M/2}}.$$

Using these estimates it is now easy to see that $\lim_{N\to\infty}\sup_{x\in\mathbf{R}^n}|D_N(x)| = 0$. □

Next we have the following important result regarding distributions with compact support:

**Theorem 2.3.21.** *If $u$ is in $\mathscr{E}'(\mathbf{R}^n)$, then $\widehat{u}$ is a real analytic function on $\mathbf{R}^n$. In particular, $\widehat{u}$ is a $\mathscr{C}^\infty$ function. Furthermore, $\widehat{u}$ and all of its derivatives have polynomial growth at infinity. Moreover, $\widehat{u}$ has a holomorphic extension on $\mathbf{C}^n$.*

*Proof.* Given a distribution $u$ with compact support and a polynomial $p(\xi)$, the action of $u$ on the $\mathscr{C}^\infty$ function $\xi \mapsto p(\xi)e^{-2\pi i x\cdot\xi}$ is a well defined function of $x$, which we denote by $u(p(\cdot)e^{-2\pi i x\cdot(\cdot)})$. Here $x$ is an element of $\mathbf{R}^n$ but the same assertion is valid if $x=(x_1,\dots,x_n)\in\mathbf{R}^n$ is replaced by $z=(z_1,\dots,z_n)\in\mathbf{C}^n$. In this case we define the dot product of $\xi$ and $z$ via $\xi\cdot z=\sum_{k=1}^n \xi_k z_k$.

It is straightforward to verify that the function of $z=(z_1,\dots,z_n)$

$$F(z) = u\big(e^{-2\pi i(\cdot)\cdot z}\big)$$

defined on $\mathbf{C}^n$ is holomorphic, in fact entire. Indeed, the continuity and linearity of $u$ and the fact that $(e^{-2\pi i\xi_j h}-1)/h \to -2\pi i\xi_j$ in $\mathscr{C}^\infty(\mathbf{R}^n)$ as $h\to 0$, $h\in\mathbf{C}$, imply that $F$ is holomorphic in every variable and its derivative with respect to $z_j$ is the action of the distribution $u$ to the $\mathscr{C}^\infty$ function

$$\xi \mapsto (-2\pi i\xi_j)e^{-2\pi i\sum_{j=1}^n \xi_j z_j}.$$

By induction it follows that for all multi-indices $\alpha$ we have

$$\partial_{z_1}^{\alpha_1}\cdots\partial_{z_n}^{\alpha_n} F = u\big((-2\pi i(\cdot))^\alpha e^{-2\pi i\sum_{j=1}^n(\cdot)z_j}\big).$$

Since $F$ is entire, its restriction on $\mathbf{R}^n$, i.e., $F(x_1,\dots,x_n)$, where $x_j=\operatorname{Re} z_j$, is real analytic. Also, an easy calculation using (2.3.4) and Leibniz's rule yield that the restriction of $F$ on $\mathbf{R}^n$ and all of its derivatives have polynomial growth at infinity.

Now for $f$ in $\mathscr{S}(\mathbf{R}^n)$ we have

$$\langle\widehat{u},f\rangle = \langle u,\widehat{f}\rangle = u\left(\int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx\right) = \int_{\mathbf{R}^n} f(x)u(e^{-2\pi i x\cdot(\cdot)})\,dx,$$

provided we can justify the passage of $u$ inside the integral. The reason for this is that the Riemann sums of the integral of $f(x)e^{-2\pi i x\cdot\xi}$ over $\mathbf{R}^n$ converge to it in the topology of $\mathscr{C}^\infty$, and thus the linear functional $u$ can be interchanged with the integral. We conclude that the tempered distribution $\widehat{u}$ can be identified with the real analytic function $x\mapsto F(x)$ whose derivatives have polynomial growth at infinity.