Theorem 2.3.20. *If $u \in \mathscr{S}'$ and $\varphi \in \mathscr{S}$, then $\varphi * u$ is a $\mathscr{C}^\infty$ function and*

$$(\varphi * u)(x) = \langle u, \tau^x \widetilde{\varphi} \rangle$$

for all $x \in \mathbf{R}^n$. Moreover, there is a positive constant m and for all multi-indices α there exists a constant C_α such that

$$|\partial^\alpha (\varphi * u)(x)| \le C_\alpha (1+|x|)^m .$$

Furthermore, if u has compact support, then $\varphi * u$ is a Schwartz function.

Proof. Let ψ be in $\mathscr{S}(\mathbf{R}^n)$. We have

$$\begin{aligned} \big\langle \varphi * u, \psi \big\rangle &= \big\langle u, \widetilde{\varphi} * \psi \big\rangle \\ &= u\left(\int_{\mathbf{R}^n} \widetilde{\varphi}(\cdot - y)\psi(y)\,dy \right) \\ &= u\left(\int_{\mathbf{R}^n} (\tau^y \widetilde{\varphi})(\cdot)\psi(y)\,dy \right) \\ &= \int_{\mathbf{R}^n} \big\langle u, \tau^y \widetilde{\varphi} \big\rangle \psi(y)\,dy, \end{aligned} \tag{2.3.20}$$

where the last step is justified by the continuity of u and by the fact that the Riemann sums of the inner integral in (2.3.20) converge to that integral in the topology of $\mathscr{S}$, a fact that will be justified later. This calculation identifies $\varphi * u$ with the function

$$(\varphi * u)(x) = \big\langle u, \tau^x \widetilde{\varphi} \big\rangle . \tag{2.3.21}$$

We now show that $(\varphi * u)(x)$ is a $\mathscr{C}^\infty$ function. Let $e_j = (0,\dots,1,\dots,0)$ with 1 in the jth entry and zero elsewhere. Then

$$\frac{\tau^{-he_j}(\varphi * u)(x) - (\varphi * u)(x)}{h} = \left\langle u, \frac{\tau^{he_j}\tau^x\widetilde{\varphi} - \tau^x\widetilde{\varphi}}{h} \right\rangle \to \big\langle u, -\partial_j \tau^x \widetilde{\varphi} \big\rangle$$

by the continuity of u and the fact that $\big(\tau^{he_j}(\tau^x\widetilde{\varphi}) - \tau^x\widetilde{\varphi}\big)/h$ tends to $-\partial_j \tau^x \widetilde{\varphi} = \tau^x(\widetilde{\partial_j \varphi})$ in $\mathscr{S}$ as $h \to 0$; see Exercise 2.3.5 (a). This gives $\partial_j(\varphi * u) = \partial_j \varphi * u$ and a similar calculation for higher-order derivatives shows that $\varphi * u \in \mathscr{C}^\infty$ and that $\partial^\gamma(\varphi * u) = (\partial^\gamma \varphi) * u$ for all multi-indices γ. It follows from (2.3.3) that for some C, m, and k we have

$$\begin{aligned} |\partial^\alpha(\varphi * u)(x)| &\le C \sum_{\substack{|\gamma| \le m \\ |\beta| \le k}} \sup_{y \in \mathbf{R}^n} |y^\gamma \tau^x(\partial^{\alpha+\beta}\widetilde{\varphi})(y)| \\ &= C \sum_{\substack{|\gamma| \le m \\ |\beta| \le k}} \sup_{y \in \mathbf{R}^n} |(x+y)^\gamma (\partial^{\alpha+\beta}\widetilde{\varphi})(y)| \\ &\le C_m \sum_{|\beta| \le k} \sup_{y \in \mathbf{R}^n} (1+|x|)^m (1+|y|)^m |(\partial^{\alpha+\beta}\widetilde{\varphi})(y)| , \end{aligned} \tag{2.3.22}$$

and this clearly implies the claimed polynomial growth of $\partial^\alpha(\varphi * u)$ at infinity.

We now indicate why $\varphi * u$ is Schwartz whenever u has compact support. Applying estimate (2.3.4) to the function $y \mapsto \varphi(x-y)$ yields that

$$\big|\langle u, \varphi(x-\cdot)\rangle\big| = |(\varphi * u)(x)| \le C \sum_{|\alpha|\le m} \sup_{|y|\le N} |\partial_y^\alpha \varphi(x-y)|$$

for some constants C, m, N. Since for $|x| \ge 2N$ we have

$$|\partial_y^\alpha \varphi(x-y)| \le C_{\alpha,M}(1+|x-y|)^{-M} \le C_{\alpha,M,N}(1+|x|)^{-M},$$

it follows that $\varphi * u$ decays rapidly at infinity. Since $\partial^\gamma(\varphi * u) = (\partial^\gamma \varphi) * u$, the same argument yields that all the derivatives of $\varphi * u$ decay rapidly at infinity; hence $\varphi * u$ is a Schwartz function. Incidentally, this argument actually shows that any Schwartz seminorm of $\varphi * u$ is controlled by a finite sum of Schwartz seminorms of φ.

We now return to the point left open concerning the convergence of the Riemann sums in (2.3.20) in the topology of $\mathscr{S}(\mathbf{R}^n)$. For each $N = 1, 2, \dots$, consider a partition of $[-N,N]^n$ into $(2N^2)^n$ cubes Q_m of side length $1/N$ and let y_m be the center of each Q_m. For multi-indices α, β, we must show that

$$D_N(x) = \sum_{m=1}^{(2N^2)^n} x^\alpha \partial_x^\beta \widetilde{\varphi}(x-y_m)\psi(y_m)|Q_m| - \int_{\mathbf{R}^n} x^\alpha \partial_x^\beta \widetilde{\varphi}(x-y)\psi(y)\,dy$$

converges to zero in $L^\infty(\mathbf{R}^n)$ as $N \to \infty$. We have

$$\begin{aligned} x^\alpha \partial_x^\beta \widetilde{\varphi}(x-y_m)\psi(y_m)|Q_m| &- \int_{Q_m} x^\alpha \partial_x^\beta \widetilde{\varphi}(x-y)\psi(y)\,dy \\ &= \int_{Q_m} x^\alpha (y-y_m)\cdot \nabla\big(\partial_x^\beta \widetilde{\varphi}(x-\cdot)\psi\big)(\xi)\,dy \end{aligned}$$

for some $\xi = y + \theta(y_m - y)$, where $\theta \in [0,1]$. Distributing the gradient to both factors, we see that the last integrand is at most

$$C\,|x|^{|\alpha|}\,\frac{\sqrt{n}}{N}\,\frac{1}{(1+|x-\xi|)^{M/2}}\,\frac{1}{(2+|\xi|)^M}$$

for M large (pick $M > 2\max\{|\alpha|, n\}$), which in turn is at most

$$C'\,|x|^{|\alpha|}\,\frac{\sqrt{n}}{N}\,\frac{1}{(1+|x|)^{M/2}}\,\frac{1}{(2+|\xi|)^{M/2}} \le C'\,|x|^{|\alpha|}\,\frac{\sqrt{n}}{N}\,\frac{1}{(1+|x|)^{M/2}}\,\frac{1}{(1+|y|)^{M/2}},$$

since $|y| \le |\xi| + \theta|y-y_m| \le |\xi| + \sqrt{n}/N \le |\xi|+1$ for $N \ge \sqrt{n}$. Inserting the estimate obtained for the integrand in the last displayed integral, we obtain

$$|D_N(x)| \le \frac{C''}{N}\,\frac{|x|^{|\alpha|}}{(1+|x|)^{M/2}} \int_{[-N,N]^n} \frac{dy}{(1+|y|)^{M/2}} + \int_{([-N,N]^n)^c} |x^\alpha \partial_x^\beta \widetilde{\varphi}(x-y)\psi(y)|\,dy.$$