It is easy to see that the operations of translation, dilation, reflection, and differentiation are continuous on tempered distributions.

Example 2.3.12. The Dirac mass at the origin δ_0 is equal to its reflection, while $\delta^a \delta_0 = a^{-n}\delta_0$ for $a > 0$. Also, $\tau^x \delta_0 = \delta_x$ for any $x \in \mathbf{R}^n$.

Now observe that for f, g, and h in $\mathscr{S}$ we have

$$\int_{\mathbf{R}^n} (h*g)(x)f(x)\,dx = \int_{\mathbf{R}^n} g(x)(\widetilde{h}*f)(x)\,dx\,. \tag{2.3.14}$$

Motivated by (2.3.14), we define the convolution of a function with a tempered distribution as follows:

Definition 2.3.13. Let $u \in \mathscr{S}'$ and $h \in \mathscr{S}$. Define the convolution $h*u$ by

$$\langle h*u, f\rangle = \langle u, \widetilde{h}*f\rangle, \qquad f \in \mathscr{S}. \tag{2.3.15}$$

Example 2.3.14. Let $u = \delta_{x_0}$ and $f \in \mathscr{S}$. Then $f*\delta_{x_0}$ is the function $x \mapsto f(x-x_0)$, for when $h \in \mathscr{S}$, we have

$$\langle f*\delta_{x_0}, h\rangle = \langle \delta_{x_0}, \widetilde{f}*h\rangle = (\widetilde{f}*h)(x_0) = \int_{\mathbf{R}^n} f(x-x_0)h(x)\,dx\,.$$

It follows that convolution with δ_0 is the identity operator.

We now define the product of a function and a distribution.

Definition 2.3.15. Let $u \in \mathscr{S}'$ and let h be a $\mathscr{C}^\infty$ function that has at most polynomial growth at infinity and the same is true for all of its derivatives. This means that for all α it satisfies $|(\partial^\alpha h)(x)| \le C_\alpha(1+|x|)^{k_\alpha}$ for some $C_\alpha, k_\alpha > 0$. Then define the product hu of h and u by

$$\langle hu, f\rangle = \langle u, hf\rangle, \qquad f \in \mathscr{S}. \tag{2.3.16}$$

Note that hf is in $\mathscr{S}$ and thus (2.3.16) is well defined. The product of an arbitrary $\mathscr{C}^\infty$ function with a tempered distribution is not defined.

We observe that if a function g is supported in a set K, then for all $f \in \mathscr{C}_0^\infty(K^c)$ we have

$$\int_{\mathbf{R}^n} f(x)g(x)\,dx = 0\,. \tag{2.3.17}$$

Moreover, the support of g is the intersection of all closed sets K with the property (2.3.17) for all f in $\mathscr{C}_0^\infty(K^c)$. Motivated by the preceding observation we give the following:

Definition 2.3.16. Let u be in $\mathscr{D}'(\mathbf{R}^n)$. The *support* of u ($\operatorname{supp} u$) is the intersection of all closed sets K with the property

$$\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^n), \qquad \operatorname{supp}\varphi \subseteq \mathbf{R}^n \setminus K \implies \langle u, \varphi\rangle = 0\,. \tag{2.3.18}$$