integrate by parts, apply the Cauchy–Schwarz inequality, Plancherel's identity, and the identity $\sum_{j=1}^{n} |\widehat{\partial_j f}(\xi)|^2 = 4\pi^2|\xi|^2|\widehat{f}(\xi)|^2$ for all $\xi \in \mathbf{R}^n$. Then replace $f(x)$ by $f(x)e^{2\pi i x\cdot z}$.$\big]$

**2.2.14.** Let $-\infty < \alpha < \frac{n}{2} < \beta < +\infty$. Prove the validity of the following inequality:

$$\big\|g\big\|_{L^1(\mathbf{R}^n)} \le C\,\big\||x|^{\alpha} g(x)\big\|_{L^2(\mathbf{R}^n)}^{\frac{\beta-n/2}{\beta-\alpha}}\big\||x|^{\beta} g(x)\big\|_{L^2(\mathbf{R}^n)}^{\frac{n/2-\alpha}{\beta-\alpha}}$$

for some constant $C = C(n,\alpha,\beta)$ independent of $g$.
$\big[$*Hint:* First prove $\|g\|_{L^1} \le C\|\,|x|^{\alpha} g(x)\|_{L^2} + \|\,|x|^{\beta} g(x)\|_{L^2}$ and then replace $g(x)$ by $g(\lambda x)$ for some suitable $\lambda > 0$.$\big]$

## 2.3 The Class of Tempered Distributions

The fundamental idea of the theory of distributions is that it is generally easier to work with linear functionals acting on spaces of "nice" functions than to work with "bad" functions directly. The set of "nice" functions we consider is closed under the basic operations in analysis, and these operations are extended to distributions by duality. This wonderful interpretation has proved to be an indispensable tool that has clarified many situations in analysis.

### *2.3.1 Spaces of Test Functions*

We recall the space $\mathscr{C}_0^\infty(\mathbf{R}^n)$ of all smooth functions with compact support, and $\mathscr{C}^\infty(\mathbf{R}^n)$ of all smooth functions on $\mathbf{R}^n$. We are mainly interested in the three spaces of "nice" functions on $\mathbf{R}^n$ that are nested as follows:

$$\mathscr{C}_0^\infty(\mathbf{R}^n) \subseteq \mathscr{S}(\mathbf{R}^n) \subseteq \mathscr{C}^\infty(\mathbf{R}^n)\,.$$

Here $\mathscr{S}(\mathbf{R}^n)$ is the space of Schwartz functions introduced in Section 2.2.

**Definition 2.3.1.** We define convergence of sequences in these spaces. We say that

$$\begin{aligned}
f_k \to f \text{ in } \mathscr{C}^\infty \quad &\iff \quad f_k, f \in \mathscr{C}^\infty \text{ and } \lim_{k\to\infty} \sup_{|x|\le N} |\partial^\alpha (f_k - f)(x)| = 0 \\
&\qquad\ \forall\ \alpha \text{ multi-indices and all } N = 1,2,\dots . \\
f_k \to f \text{ in } \mathscr{S} \quad &\iff \quad f_k, f \in \mathscr{S} \text{ and } \lim_{k\to\infty} \sup_{x\in\mathbf{R}^n} |x^\alpha \partial^\beta (f_k - f)(x)| = 0 \\
&\qquad\ \forall\ \alpha,\beta \text{ multi-indices.} \\
f_k \to f \text{ in } \mathscr{C}_0^\infty \quad &\iff \quad f_k, f \in \mathscr{C}_0^\infty,\ \exists\, B \text{ compact}, \text{ support}(f_k) \subseteq B \text{ for all } k, \\
&\qquad\ \text{and } \lim_{k\to\infty} \big\|\partial^\alpha (f_k - f)\big\|_{L^\infty} = 0\ \forall\ \alpha \text{ multi-indices.}
\end{aligned}$$