2.2.10. Let f be in $L^1(\mathbf{R})$. Prove that

$$\int_{-\infty}^{+\infty} f\Big(x - \frac{1}{x}\Big)\, dx = \int_{-\infty}^{+\infty} f(u)\, du\,.$$

$\big[$*Hint:* For $x \in (-\infty, 0)$ use the change of variables $u = x - \frac{1}{x}$ or $x = \frac{1}{2}\big(u - \sqrt{4+u^2}\big)$. For $x \in (0,\infty)$ use the change of variables $u = x - \frac{1}{x}$ or $x = \frac{1}{2}\big(u + \sqrt{4+u^2}\big)$.$\big]$

2.2.11. (a) Use Exercise 2.2.10 with $f(x) = e^{-tx^2}$ to obtain the *subordination* identity

$$e^{-2t} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y - t^2/y} \frac{dy}{\sqrt{y}}\,, \qquad \text{where } t > 0.$$

(b) Set $t = \pi|x|$ and integrate with respect to $e^{-2\pi i \xi\cdot x}dx$ to prove that

$$(e^{-2\pi|x|})\widehat{\ }(\xi) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}\,.$$

This calculation gives the Fourier transform of the Poisson kernel.

2.2.12. Let $1 \le p \le \infty$ and let p' be its dual index.
(a) Prove that Schwartz functions f on the line satisfy the estimate

$$\|f\|_{L^\infty}^2 \le 2\|f\|_{L^p}\|f'\|_{L^{p'}}\,.$$

(b) Prove that all Schwartz functions f on $\mathbf{R}^n$ satisfy the estimate

$$\|f\|_{L^\infty}^2 \le \sum_{\alpha+\beta=(1,\dots,1)} \|\partial^\alpha f\|_{L^p} \|\partial^\beta f\|_{L^{p'}}\,,$$

where the sum is taken over all pairs of multi-indices α and β satisfying $\alpha_j + \beta_j = 1$ for all $j = 1, 2, \dots, n$.
$\big[$*Hint:* Part (a): Write $f(x)^2 = \int_{-\infty}^x \frac{d}{dt} f(t)^2\, dt$.$\big]$

2.2.13. The *uncertainty principle* says that the position and the momentum of a particle cannot be simultaneously localized. Prove the following inequality, which presents a quantitative version of this principle:

$$\|f\|_{L^2(\mathbf{R}^n)}^2 \le \frac{4\pi}{n} \inf_{y\in\mathbf{R}^n} \Big[\int_{\mathbf{R}^n} |x-y|^2 |f(x)|^2\, dx\Big]^{\frac12} \inf_{z\in\mathbf{R}^n} \Big[\int_{\mathbf{R}^n} |\xi - z|^2 |\widehat{f}(\xi)|^2\, d\xi\Big]^{\frac12},$$

where f is a Schwartz function on $\mathbf{R}^n$ (or an L^2 function with sufficient decay at infinity).
$\big[$*Hint:* Let y be in $\mathbf{R}^n$. Start with

$$\|f\|_{L^2}^2 = \frac{1}{n} \int_{\mathbf{R}^n} f(x)\overline{f(x)} \sum_{j=1}^n \frac{\partial}{\partial x_j}(x_j - y_j)\, dx\,,$$