*Proof.* Consider the function $\chi_{[a,b]}$ on $\mathbf{R}$. A simple computation gives

$$\widehat{\chi_{[a,b]}}(\xi) = \int_a^b e^{-2\pi i x \xi}\,dx = \frac{e^{-2\pi i \xi a} - e^{-2\pi i \xi b}}{2\pi i \xi},$$

which tends to zero as $|\xi| \to \infty$. Likewise, if $g = \prod_{j=1}^n \chi_{[a_j,b_j]}$ on $\mathbf{R}^n$, then

$$\widehat{g}(\xi) = \prod_{j=1}^n \frac{e^{-2\pi i \xi_j a_j} - e^{-2\pi i \xi_j b_j}}{2\pi i \xi_j},$$

with the understanding that if $\xi_j = 0$ for some $j$, then the corresponding factor is $b_j - a_j$. Given a $\xi = (\xi_1, \dots, \xi_n) \neq 0$, there is $j_0$ such that $|\xi_{j_0}| \geq |\xi|/\sqrt{n}$. Then

$$\left| \prod_{j=1}^n \frac{e^{-2\pi i \xi_j a_j} - e^{-2\pi i \xi_j b_j}}{2\pi i \xi_j} \right| \leq \frac{2\sqrt{n}}{2\pi |\xi|} \sup_{1 \leq j_0 \leq n} \prod_{j \neq j_0} (b_j - a_j)$$

which also tends to zero as $|\xi| \to \infty$ in $\mathbf{R}^n$. Here we used that $|e^{ix} - e^{iy}| \leq |x-y|$.

Given a general integrable function $f$ on $\mathbf{R}^n$ and $\varepsilon > 0$, there is a simple function $h$, which is a finite linear combination of characteristic functions of rectangles (like $g$), such that $\|f-h\|_{L^1} < \frac{\varepsilon}{2}$. Then there is an $M$ is such that for $|\xi| > M$ we have $|\widehat{h}(\xi)| < \frac{\varepsilon}{2}$. It follows that

$$|\widehat{f}(\xi)| \leq |\widehat{f}(\xi) - \widehat{h}(\xi)| + |\widehat{h}(\xi)| \leq \big\|f-h\big\|_{L^1} + |\widehat{h}(\xi)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

provided $|\xi| > M$. This implies that $|\widehat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.

A different proof can be given by taking the function $h$ in the preceding paragraph to be a Schwartz function and using that Schwartz functions are dense in $L^1(\mathbf{R}^n)$; see Exercise 2.2.5 about the last assertion. □

We end this section with an example that illustrates some of the practical uses of the Fourier transform.

**Example 2.2.18.** We would like to find a Schwartz function $f(x_1,x_2,x_3)$ on $\mathbf{R}^3$ that satisfies the partial differential equation

$$f(x) + \partial_1^2 \partial_2^2 \partial_3^4 f(x) + 4i\partial_1^2 f(x) + \partial_2^7 f(x) = e^{-\pi|x|^2}.$$

Taking the Fourier transform on both sides of this identity and using Proposition 2.2.11 (2), (9) and the result of Example 2.2.9, we obtain

$$\widehat{f}(\xi)\Big[1 + (2\pi i \xi_1)^2 (2\pi i \xi_2)^2 (2\pi i \xi_3)^4 + 4i(2\pi i \xi_1)^2 + (2\pi i \xi_2)^7\Big] = e^{-\pi|\xi|^2}.$$

Let $p(\xi) = p(\xi_1,\xi_2,\xi_3)$ be the polynomial inside the square brackets. We observe that $p(\xi)$ has no real zeros and we may therefore write

$$\widehat{f}(\xi) = e^{-\pi|\xi|^2} p(\xi)^{-1} \implies f(x) = \big(e^{-\pi|\xi|^2} p(\xi)^{-1}\big)^{\vee}(x).$$