on $\mathbf{S}^{n-1}$; this minimum is positive since this function has no zeros on $\mathbf{S}^{n-1}$. A related inequality is

$$(1+|x|)^k \le 2^k(1+C_{n,k}) \sum_{|\beta|\le k} |x^\beta| \,. \tag{2.2.3}$$

This follows from (2.2.2) for $|x|\ge 1$, while for $|x|<1$ we note that the sum in (2.2.3) is at least one since $|x^{(0,\dots,0)}|=1$.

We end the preliminaries by noting the validity of the one-dimensional Leibniz rule

$$\frac{d^m}{dt^m}(fg) = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dt^k}\,\frac{d^{m-k} g}{dt^{m-k}}\,, \tag{2.2.4}$$

for all $\mathscr{C}^m$ functions f,g on $\mathbf{R}$, and its multidimensional analogue

$$\partial^\alpha(fg) = \sum_{\beta\le\alpha} \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n} (\partial^\beta f)(\partial^{\alpha-\beta} g)\,, \tag{2.2.5}$$

for f,g in $\mathscr{C}^{|\alpha|}(\mathbf{R}^n)$ for some multi-index α, where the notation $\beta\le\alpha$ in (2.2.5) means that β ranges over all multi-indices satisfying $0\le\beta_j\le\alpha_j$ for all $1\le j\le n$. We observe that identity (2.2.5) is easily deduced by repeated application of (2.2.4), which in turn is obtained by induction.

2.2.1 The Class of Schwartz Functions

We now introduce the class of *Schwartz functions* on $\mathbf{R}^n$. Roughly speaking, a function is Schwartz if it is smooth and all of its derivatives decay faster than the reciprocal of any polynomial at infinity. More precisely, we give the following definition.

Definition 2.2.1. A $\mathscr{C}^\infty$ complex-valued function f on $\mathbf{R}^n$ is called a Schwartz function if for every pair of multi-indices α and β there exists a positive constant $C_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta}(f) = \sup_{x\in\mathbf{R}^n} |x^\alpha \partial^\beta f(x)| = C_{\alpha,\beta} < \infty. \tag{2.2.6}$$

The quantities $\rho_{\alpha,\beta}(f)$ are called the *Schwartz seminorms* of f. The set of all Schwartz functions on $\mathbf{R}^n$ is denoted by $\mathscr{S}(\mathbf{R}^n)$.

Example 2.2.2. The function $e^{-|x|^2}$ is in $\mathscr{S}(\mathbf{R}^n)$ but $e^{-|x|}$ is not, since it fails to be differentiable at the origin. The $\mathscr{C}^\infty$ function $g(x)=(1+|x|^4)^{-a}$, $a>0$, is not in $\mathscr{S}$ since it does not decay faster than the reciprocal of a fixed polynomial at infinity. The set of all smooth functions with compact support, $\mathscr{C}_0^\infty(\mathbf{R}^n)$, is contained in $\mathscr{S}(\mathbf{R}^n)$.

Remark 2.2.3. If f_1 is in $\mathscr{S}(\mathbf{R}^n)$ and f_2 is in $\mathscr{S}(\mathbf{R}^m)$, then the function of $m+n$ variables $f_1(x_1,\dots,x_n)f_2(x_{n+1},\dots,x_{n+m})$ is in $\mathscr{S}(\mathbf{R}^{n+m})$. If f is in $\mathscr{S}(\mathbf{R}^n)$ and $P(x)$ is a polynomial of n variables, then $P(x)f(x)$ is also in $\mathscr{S}(\mathbf{R}^n)$. If α is a multi-index and f is in $\mathscr{S}(\mathbf{R}^n)$, then $\partial^\alpha f$ is in $\mathscr{S}(\mathbf{R}^n)$. Also note that