(d) Prove that $M = \max(M_R, M_L)$.
(e) Let $N = \min(M_R, M_L)$. Obtain the following consequence of part (a),

$$\int_{\mathbf{R}} M(f)^p + N(f)^p \, dx = \frac{p}{p-1} \int_{\mathbf{R}} |f| \left(M(f)^{p-1} + N(f)^{p-1} \right) dx,$$

(f) Use part (e) to prove that

$$(p-1)\big\|M(f)\big\|_{L^p}^p - p\big\|f\big\|_{L^p}\big\|M(f)\big\|_{L^p}^{p-1} - \big\|f\big\|_{L^p}^p \le 0.$$

$\big[$*Hint:* (a) Write the set $E_\alpha = \{M_R(f) > \alpha\}$ as a union of open intervals (a_j, b_j). For each x in (a_j, b_j), let $N_x = \big\{s \in \mathbf{R} : \int_x^s |f| > \alpha(s-x)\big\} \cap (x, b_j]$. Show that N_x is nonempty and that $\sup N_x = b_j$ for every $x \in (a_j, b_j)$. Conclude that $\int_{a_j}^{b_j} |f(t)|\, dt \ge \alpha(b_j - a_j)$, which implies that each a_j is finite. For the reverse inequality use that $a_j \notin E_\alpha$. Part (d) is due to K. L. Phillips. (e) First obtain a version of the equality with M_R in the place of M and M_L in the place of N. Then use that $M(f)^q + N(f)^q = M_L(f)^q + M_R(f)^q$ for all q. (f) Use that $|f|N(f)^{p-1} \le \frac{1}{p}|f|^p + \frac{1}{p'}N(f)^p$. This alternative proof of the result in Exercise 2.1.2(c) was suggested by J. Duoandikoetxea.$\big]$

2.1.12. A cube $Q = [a_1 2^k, (a_1+1)2^k) \times \cdots \times [a_n 2^k, (a_n+1)2^k)$ on $\mathbf{R}^n$ is called *dyadic* if $k, a_1, \dots, a_n \in \mathbf{Z}$. Observe that either two dyadic cubes are disjoint or one contains the other. Define the *dyadic maximal function*

$$M_d(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all dyadic cubes Q containing x.
(a) Prove that M_d maps L^1 to $L^{1,\infty}$ with constant at most one. Presicely, show that for all $\alpha > 0$ and $f \in L^1(\mathbf{R}^n)$ we have

$$|\{x \in \mathbf{R}^n : \, M_d(f)(x) > \alpha\}| \le \frac{1}{\alpha} \int_{\{M_d(f) > \alpha\}} |f(t)| \, dt.$$

(b) Conclude that M_d maps $L^p(\mathbf{R}^n)$ to itself with constant at most $p/(p-1)$.

2.1.13. Observe that the proof of Theorem 2.1.6 yields the estimate

$$\lambda |\{M(f) > \lambda\}|^{\frac{1}{p}} \le 3^n |\{M(f) > \lambda\}|^{-1+\frac{1}{p}} \int_{\{M(f) > \lambda\}} |f(y)| \, dy$$

for $\lambda > 0$ and f locally integrable. Use the result of Exercise 1.1.12(a) to prove that the Hardy–Littlewood maximal operator M maps the space $L^{p,\infty}(\mathbf{R}^n)$ to itself for $1 < p < \infty$.