$$\lim_{\substack{|B|\to 0\\ B\ni x}} \frac{1}{|B|}\int_B |f(y)-f(x)|^p\,dy = 0\,,$$

where the limit is taken over all open balls B containing x and shrinking to $\{x\}$.
(c) Conclude that for any f in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and for almost all $x\in\mathbf{R}^n$ we have

$$\lim_{\substack{|B|\to 0\\ B\ni x}} \frac{1}{|B|}\int_B f(y)\,dy = f(x)\,,$$

where the limit is taken over all open balls B containing x and shrinking to $\{x\}$.
$\big[$*Hint:* (a) Define an oscillation $O_f(y)=\limsup_{\varepsilon\to 0}|T_\varepsilon(f)(y)|$. For all f in $L^p(X)$ and $g\in D$ we have that $O_f(y)\le KO_{f-g}(y)$. Then use the argument in the proof of Theorem 2.1.14. (b) Apply part (a) with

$$T_\varepsilon(f)(x) = \sup_{B(z,\varepsilon)\ni x}\left(\frac{1}{|B(z,\varepsilon)|}\int_{B(z,\varepsilon)}|f(y)-f(x)|^p\,dy\right)^{1/p},$$

observing that $T_*(f)=\sup_{\varepsilon>0}T_\varepsilon(f)\le \max(1,2^{\frac{1-p}{p}})\big(|f|+M(|f|^p)^{\frac1p}\big)$. (c) Follows from part (b) with $p=1$. Note that part (b) can be proved without part (a) but using part (c) as follows: for every rational number a there is a set E_a of Lebesgue measure zero such that for $x\in\mathbf{R}^n\setminus E_a$ we have $\lim_{B\ni x,|B|\to 0}\frac{1}{|B|}\int_B|g(y)-a|^p\,dy=|g(x)-a|^p$, since the function $y\mapsto|f(y)-a|^p$ is in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. By considering an enumeration of the rationals, find a set of measure zero E such for $x\notin E$ the preceding limit exists for all rationals a and by continuity for all real numbers a, in particular for $a=g(x)$.$\big]$

2.1.11. Let f be in $L^1(\mathbf{R})$. Define the right maximal function $M_R(f)$ and the left maximal function $M_L(f)$ as follows:

$$\begin{aligned} M_L(f)(x) &= \sup_{r>0}\frac{1}{r}\int_{x-r}^{x}|f(t)|\,dt\,,\\ M_R(f)(x) &= \sup_{r>0}\frac{1}{r}\int_{x}^{x+r}|f(t)|\,dt\,. \end{aligned}$$

(a) Show that for all $\alpha>0$ and $f\in L^1(\mathbf{R})$ we have

$$\begin{aligned} |\{x\in\mathbf{R}:\, M_L(f)(x)>\alpha\}| &= \frac{1}{\alpha}\int_{\{M_L(f)>\alpha\}}|f(t)|\,dt\,,\\ |\{x\in\mathbf{R}:\, M_R(f)(x)>\alpha\}| &= \frac{1}{\alpha}\int_{\{M_R(f)>\alpha\}}|f(t)|\,dt\,. \end{aligned}$$

(b) Extend the definition of $M_L(f)$ and $M_R(f)$ for $f\in L^p(\mathbf{R})$ for $1\le p\le\infty$. Show that M_L and M_R map L^p to L^p with norm at most $p/(p-1)$ for all p with $1<p<\infty$.
(c) Construct examples to show that the operator norms of M_L and M_R on $L^p(\mathbf{R})$ are exactly $p/(p-1)$ for $1<p<\infty$.