2.1.6. Let $M_s(f)(x)$ be the supremum of the averages of $|f|$ over all rectangles with sides parallel to the axes containing x. The operator M_s is called the *strong maximal function*.
(a) Prove that M_s maps $L^p(\mathbf{R}^n)$ to itself when $1 < p < \infty$.
(b) Show that the operator norm of M_s is A_p^n, where A_p is as in Exercise 2.1.2 (c).
(c) Prove that M_s is not weak type $(1,1)$.

2.1.7. Prove that if

$$|\varphi(x_1,\dots,x_n)| \le A(1+|x_1|)^{-1-\varepsilon}\cdots(1+|x_n|)^{-1-\varepsilon}$$

for some $A, \varepsilon > 0$, and $\varphi_{t_1,\dots,t_n}(x) = t_1^{-1}\cdots t_n^{-1}\varphi(t_1^{-1}x_1,\dots,t_n^{-1}x_n)$, then the maximal operator

$$f \mapsto \sup_{t_1,\dots,t_n>0}|f * \varphi_{t_1,\dots,t_n}|$$

is pointwise controlled by the strong maximal function.

2.1.8. Prove that for any fixed $1 < p < \infty$, the operator norm of M on $L^p(\mathbf{R}^n)$ tends to infinity as $n \to \infty$.
$\left[\textit{Hint:} \text{ Let } f_0 \text{ be the characteristic function of the unit ball in } \mathbf{R}^n. \text{ Consider the averages } |B_x|^{-1}\int_{B_x} f_0\,dy, \text{ where } B_x = B\big(\frac{1}{2}(|x|-|x|^{-1})\frac{x}{|x|}, \frac{1}{2}(|x|+|x|^{-1})\big) \text{ for } |x| > 1.\right]$

2.1.9. (a) In $\mathbf{R}^2$ let $M_0(f)(x)$ be the maximal function obtained by taking the supremum of the averages of $|f|$ over all rectangles (of arbitrary orientation) containing x. Prove that M_0 is not bounded on $L^p(\mathbf{R}^n)$ for $p \le 2$ and conclude that M_0 is not weak type $(1,1)$.
(b) Let $M_{00}(f)(x)$ be the maximal function obtained by taking the supremum of the averages of $|f|$ over all rectangles in $\mathbf{R}^2$ of arbitrary orientation but fixed eccentricity containing x. (The eccentricity of a rectangle is the ratio of its longer side to its shorter side.) Using a covering lemma, show that M_{00} is weak type $(1,1)$ with a bound proportional to the square of the eccentricity.
(c) On $\mathbf{R}^n$ define a maximal function by taking the supremum of the averages of $|f|$ over all products of intervals $I_1 \times \cdots \times I_n$ containing a point x with $|I_2| = a_2|I_1|,\dots,|I_n| = a_n|I_1|$ and $a_2,\dots,a_n > 0$ fixed. Show that this maximal function is of weak type $(1,1)$ with bound independent of the numbers $a_2,\dots,a_n$.
$\left[\textit{Hint:} \text{ Part (b): Let } b \text{ be the eccentricity. If two rectangles with the same eccentricity intersect, then the smaller one is contained in the bigger one scaled } 4b \text{ times. Then use an argument similar to that in Lemma 2.1.5.}\right]$

2.1.10. (a) Let $0 < p,q < \infty$ and let X,Y be measure spaces. Suppose that T_ε are maps from $L^p(X)$ to $L^{q,\infty}(Y)$ satisfy $|T_\varepsilon(f+g)| \le K(|T_\varepsilon(f)| + |T_\varepsilon(g)|)$ for all $\varepsilon > 0$ and all $f,g \in L^p(X)$, and also $\lim_{\varepsilon\to 0} T_\varepsilon(f) = 0$ a.e. for all f in some dense subspace D of $L^p(X)$. Assume furthermore that the maximal operator $T_*(f) = \sup_{\varepsilon>0}|T_\varepsilon(f)|$ maps $L^p(X)$ to $L^{q,\infty}(Y)$. Prove that $\lim_{\varepsilon\to 0} T_\varepsilon(f) = 0$ a.e. for all f in $L^p(X)$.
(b) Use the result in part (a) to prove the following version of the Lebesgue differentiation theorem: Let $f \in L^p(\mathbf{R}^n)$ for some $0 < p < \infty$. Then for almost all $x \in \mathbf{R}^n$ we have