(b) Deduce that if f is supported in a ball B and $f\log^+(2|f|)$ is integrable over B, then $M(f)$ is integrable over the same ball B.
(c) ([375], [336]) Apply Proposition 2.1.20 to $|f|$ and $\alpha > 0$ and Exercise 2.1.3 to show that with $c_n = 2^n(n^{n/2}v_n)^{-1}$ we have

$$|\{x\in\mathbf{R}^n:\ M(f)(x) > c_n\alpha\}| \ge \frac{2^{-n}}{\alpha}\int_{\{|f|>\alpha\}}|f(y)|\,dy\,.$$

(d) Suppose that f is integrable and supported in a ball $B(0,\rho)$. Show that for x in $B(0,2\rho)\setminus B(0,\rho)$ we have $\mathcal{M}(f)(x)\le\mathcal{M}(f)(\rho^2|x|^{-2}x)$. Conclude that

$$\int_{B(0,2\rho)}\mathcal{M}(f)\,dx \le (4^n+1)\int_{B(0,\rho)}\mathcal{M}(f)\,dx$$

and from this deduce a similar inequality for $M(f)$.
(e) Suppose that f is integrable and supported in a ball B and that $M(f)$ is integrable over B. Let $\lambda_0 = 2^n|B|^{-1}\|f\|_{L^1}$. Use parts (b) and (c) to prove that $f\log^+(\lambda_0^{-1}c_n|f|)$ is integrable over B.
$\big[$*Hint:* Part (a): Write $f = f\chi_{|f|>\alpha} + f\chi_{|f|\le\alpha}$. Part (b): Show that $M(f\chi_E)$ is integrable over B, where $E = \{|f|\ge 1/2\}$. Part (c): Use Proposition 2.1.20. Part (d): Let $x' = \rho^2|x|^{-2}x$ for some $\rho < |x| < 2\rho$. Show that for $R > |x|-\rho$, we have that

$$\int_{B(x,R)}|f(z)|\,dz \le \int_{B(x',R)}|f(z)|\,dz$$

by showing that $B(x,R)\cap B(0,\rho)\subset B(x',R)$. Part (e): For $x\notin 2B$ we have $M(f)(x)\le\lambda_0$, hence $\int_{2B}M(f)(x)\,dx \ge \int_{\lambda_0}^\infty |\{x\in 2B:\ M(f)(x)>\alpha\}|\,d\alpha.\big]$

2.1.5. (*A. Kolmogorov*) Let S be a sublinear operator that maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with norm B. Suppose that $f\in L^1(\mathbf{R}^n)$. Prove that for any set A of finite Lebesgue measure and for all $0<q<1$ we have

$$\int_A |S(f)(x)|^q\,dx \le (1-q)^{-1}B^q|A|^{1-q}\big\|f\big\|_{L^1}^q,$$

and in particular, for the Hardy–Littlewood maximal operator,

$$\int_A M(f)(x)^q\,dx \le (1-q)^{-1}3^{nq}|A|^{1-q}\big\|f\big\|_{L^1}^q\,.$$

$\big[$*Hint:* Use the identity

$$\int_A |S(f)(x)|^q\,dx = \int_0^\infty q\,\alpha^{q-1}|\{x\in A:\ S(f)(x)>\alpha\}|\,d\alpha$$

and estimate the last measure by $\min(|A|,\frac{B}{\alpha}\|f\|_{L^1}).\big]$